Approximation of eigenvalue problems in mixed form — Part 2

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Maxwell eigenvalues

Ampère and Faraday's laws: find resonance frequencies $\omega \in \mathbb{R}$ (with $\omega \neq 0$) and electromagnetic fields $(\mathbf{E}, \mathbf{H}) \neq (0, 0)$ such that

$$
\begin{aligned}
\mathbf{curl}\,\mathbf{E} &= i\omega\mu\mathbf{H} && \text{in } \Omega \\
\mathbf{curl}\,\mathbf{H} &= -i\omega\varepsilon\mathbf{E} && \text{in } \Omega \\
\mathbf{E} \times \mathbf{n} &= 0 && \text{on } \partial\Omega \\
\mathbf{H} \cdot \mathbf{n} &= 0 && \text{on } \partial\Omega
\end{aligned}
$$

$\omega \neq 0$ gives divergence conditions

$$
\begin{aligned}
\operatorname{div} \varepsilon\mathbf{E} &= 0 && \text{in } \Omega \\
\operatorname{div} \mu\mathbf{H} &= 0 && \text{on } \Omega
\end{aligned}
$$

It is then standard to eliminate one field and to obtain the **curl curl** problem

Curl curl problem: strong form

Eliminate $\mathbf{H}$ and take $\mathbf{u} = \mathbf{E}$ ($\lambda = \omega^2$)

$$\begin{cases} \mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\mathbf{u}) = \lambda\varepsilon\mathbf{u} & \text{in } \Omega \\ \operatorname{div}(\varepsilon\mathbf{u}) = 0 & \text{in } \Omega \\ \mathbf{u} \times \mathbf{n} = 0 & \text{on } \partial\Omega \end{cases}$$

Well-known and intensively studied problem. Special (*edge*) finite elements required for its approximation

For ease of presentation, we take $\mu = \varepsilon = 1$ and simple topology from now on

Standard formulation

The standard variational formulation reads

$$\mathbf{u} \in \mathbf{H}_0(\mathbf{curl}) :$$
$$\begin{cases} (\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) = \lambda(\mathbf{u}, \mathbf{v}) & \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}) \\ (\mathbf{u}, \mathbf{grad}\,\phi) = 0 & \forall \phi \in H^1_0 \end{cases}$$

The most commonly used variational formulation in based on the replacement of the divergence free constraint by the condition $\lambda \neq 0$

$$\mathbf{u} \in \mathbf{H}_0(\mathbf{curl}) :$$
$$(\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) = \lambda(\mathbf{u}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl})$$

Consequence of Helmholtz decomposition $\mathbf{u} = \mathbf{grad}\,\alpha + \mathrm{curl}\,\beta$

The kernel $\lambda = 0$ corresponds to the infinite dimensional space $\mathbf{grad}\,H^1_0$

Mixed formulations

⟨Kikuchi '89⟩

Divergence free constraint imposed via Lagrange multiplier ψ

$$\mathbf{u} \in \mathbf{H}_0(\mathbf{curl}),\ \psi \in H_0^1 :$$
$$\begin{cases} (\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) + (\mathbf{grad}\,\psi, \mathbf{v}) = \lambda(\mathbf{u}, \mathbf{v}) & \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}) \\ (\mathbf{grad}\,\phi, \mathbf{u}) = 0 & \forall \phi \in H_0^1 \end{cases}$$

⟨B–Fernandes–Gastaldi–Perugia '99⟩

Second mixed formulation ($\mathbf{H}_0(\mathrm{div}^0) = \mathbf{curl}(\mathbf{H}_0(\mathbf{curl}))$)

$$\boldsymbol{\sigma} \in \mathbf{H}_0(\mathbf{curl}),\ \mathbf{z} \in \mathbf{H}_0(\mathrm{div}^0) :$$
$$\begin{cases} (\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\mathbf{curl}\,\boldsymbol{\tau}, \mathbf{z}) = 0 & \forall \boldsymbol{\tau} \in \mathbf{H}_0(\mathbf{curl}) \\ (\mathbf{curl}\,\boldsymbol{\sigma}, \mathbf{w}) = -\lambda(\mathbf{z}, \mathbf{w}) & \forall \mathbf{w} \in \mathbf{H}_0(\mathrm{div}^0) \end{cases}$$

Eigenvalues in mixed form

The equivalence with mixed formulations allowed us to apply general theory of eigenvalue approximation in mixed form

⟨B.–Brezzi–Gastaldi ’97⟩

The main tool for the analysis (exploited for the h version) is the construction of a Fortin operator that converges to the identity in norm: *Fortid* property

⟨B.–Fernandes–Gastaldi–Perugia ’99⟩
⟨B. ’00–’01⟩

Discrete Compactness Property may also be used ⟨Kikuchi ’89⟩
⟨Monk–Demkowicz ’00⟩
⟨Caorsi–Fernandes–Raffetto ’00⟩
⟨B.–Demkowicz–Costabel ’03⟩
⟨B.–Costabel–Dauge–Demkowicz ’06⟩
⟨B.–Costabel–Dauge–Demkowicz–Hiptmair ’11⟩

The two approaches are indeed equivalent ⟨B. ’07⟩

Mixed conditions for Kikuchi formulation

[ELKER] Ellipticity in the discrete kernel

There exists $\alpha > 0$ such that

$$(\operatorname{curl} \mathbf{v}_k, \operatorname{curl} \mathbf{v}_k) \geq \alpha \|\mathbf{v}_k\|_{L^2}^2 \quad \forall \mathbf{v}_k \in K_k^d$$

[WA1] Weak approximability of $Q = H_0^{1+s}$

There exists $\omega_1(k)$ tending to zero such that

$$\sup_{\mathbf{v}_k \in K_k^d} \frac{(\mathbf{v}_k, \mathbf{grad}\, \psi)}{\|\mathbf{v}_k\|_{\mathrm{curl}}} \leq \omega_1(k) \|\psi\|_{H^1} \quad \forall \psi \in Q$$

[SA1] Strong approximability of $V_0 = \mathbf{H}_0^s(\mathbf{curl}) \cap \mathbf{H}(\mathrm{div}^0)$

There exists $\omega_2(k)$ tending to zero such that for every $\mathbf{u} \in V_0$ there exists $\mathbf{u}^I \in K_k^d$ such that

$$\|\mathbf{u} - \mathbf{u}^I\|_{\mathrm{curl}} \leq \omega_2(k) \|\mathbf{u}\|_{V_0}$$

Kikuchi solution operators: continuous…

$$\begin{cases} (\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) + (\mathbf{grad}\,p, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) & \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}) \\ (\mathbf{grad}\,q, \mathbf{u}) = 0 & \forall q \in H_0^1 \end{cases}$$

$T^{Ki} \in \mathcal{L}(L^2)$: $T^{Ki}(\mathbf{f}) = \mathbf{u}$

…and discrete one

$$\begin{cases} (\mathbf{curl}\,\mathbf{u}_k, \mathbf{curl}\,\mathbf{v}) + (\mathbf{grad}\,p_k, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) & \forall \mathbf{v} \in V_k \\ (\mathbf{grad}\,q, \mathbf{u}_k) = 0 & \forall q \in Q_k \end{cases}$$

$T_k^{Ki} \in \mathcal{L}(L^2)$: $T_k^{Ki}(\mathbf{f}) = \mathbf{u}_k$

⟨B.–Brezzi–Gastaldi ’97⟩

Theorem

If the ellipticity in the discrete kernel [ELKER], the weak approximability of Q [WA1], and the strong approximability of V_0 [SA1] are satisfied, then the following convergence in norm holds true

$$\|T^{Ki} - T_k^{Ki}\|_{\mathcal{L}(L^2)} \to 0$$

Remark

Convergence in norm allows us to use the classical Babuška–Osborn theory for eigenmode convergence

⟨Babuška–Osborn ’91⟩

Mixed conditions for second formulation

[WA2] Weak approximability of $Z^0 = \mathbf{H}_0^s(\mathbf{curl}) \cap \mathbf{H}(\mathrm{div}^0)$

There exists $\omega_3(k)$ tending to zero such that

$$(\mathrm{curl}\, \boldsymbol{\tau}_k, \mathbf{z}) \leq \omega_3(k) \|\boldsymbol{\tau}_k\|_{L^2} \|\mathbf{z}\|_{Z^0} \quad \forall \boldsymbol{\tau}_k \in K_k^c, \ \forall \mathbf{z} \in Z^0$$

[SA2] Strong approximability of $Z^0 = \mathbf{H}_0^s(\mathbf{curl}) \cap \mathbf{H}(\mathrm{div}^0)$

There exists $\omega_4(k)$ tending to zero such that for every $\mathbf{z} \in Z^0$ there exists $\mathbf{z}^I \in K_k^c$ such that

$$\|\mathbf{z} - \mathbf{z}^I\|_{L^2} \leq \omega_4(k) \|\mathbf{z}\|_{Z^0}$$

Fortin operator

$\Pi_k : V^0 \to V_k$ such that $\forall \boldsymbol{\sigma} \in V^0$

$$\begin{cases} (\mathbf{curl}(\boldsymbol{\sigma} - \Pi_k \boldsymbol{\sigma}), \mathbf{w}_k) = 0 \quad \forall \mathbf{w}_k \in Z_k \\ \|\Pi_k \boldsymbol{\sigma}\|_{\mathrm{curl}} \leq C \|\boldsymbol{\sigma}\|_{V^0} \end{cases}$$

[FORTID] Fortid property

There exists $\omega_5(k)$ tending to zero such that

$$\|\boldsymbol{\sigma} - \Pi_k \boldsymbol{\sigma}\|_{L^2} \leq \omega_5(k) \|\boldsymbol{\sigma}\|_{V^0} \quad \forall \boldsymbol{\sigma} \in V^0$$

Alternative solution operators: continuous...

$$\begin{cases} (\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\mathbf{curl}\,\boldsymbol{\tau}, \mathbf{z}) = 0 & \forall \boldsymbol{\tau} \in \mathbf{H}_0(\mathbf{curl}) \\ (\mathbf{curl}\,\boldsymbol{\sigma}, \mathbf{w}) = -(\mathbf{g}, \mathbf{w}) & \forall \mathbf{w} \in \mathbf{curl}(\mathbf{H}_0(\mathbf{curl})) \end{cases}$$

$T^{M2} \in \mathcal{L}(L^2)$: $T^{M2}(\mathbf{g}) = \mathbf{z}$

...and discrete one

$$\begin{cases} (\boldsymbol{\sigma}_k, \boldsymbol{\tau}) + (\mathbf{curl}\,\boldsymbol{\tau}, \mathbf{z}_k) = 0 & \forall \boldsymbol{\tau} \in V_k \\ (\mathbf{curl}\,\boldsymbol{\sigma}_k, \mathbf{w}) = -(\mathbf{g}, \mathbf{w}) & \forall \mathbf{w} \in Z_k \end{cases}$$

$T_k^{M2} \in \mathcal{L}(L^2)$: $T_k^{M2}(\mathbf{g}) = \mathbf{z}_k$

⟨B.–Brezzi–Gastaldi ’97⟩

Theorem

If the weak approximability of Z^0 [WA2] and the strong approximability of Z^0 [SA2] are satisfied, and if there exists a Fortin operator satisfying the Fortid property [FORTID], then the following convergence in norm holds true

$$\|T^{M2} - T_k^{M2}\|_{\mathcal{L}(L^2)} \to 0$$

Compactness properties

The space $\mathbf{H}_0(\mathbf{curl}) \cap \mathbf{H}(\mathrm{div}^0)$ is compactly embedded in L^2

Compactness can be rephrased as

Given a sequence $\{\mathbf{u}_n\} \subset \mathbf{H}_0(\mathbf{curl})$ such that

$$(\mathbf{u}_n, \mathbf{grad}\,\phi) = 0 \quad \forall \phi \in H_0^1,\ \forall n$$

If $\{\mathbf{u}_n\}$ is uniformly bounded in $\mathbf{H}_0(\mathbf{curl})$, $\|\,\mathbf{curl}\,\mathbf{u}_n\|_{L^2} \leq 1$, then there exits a subsequence (still denoted $\{\mathbf{u}_n\}$) and $\mathbf{u} \in L^2$ such that

$$\|\mathbf{u}_n - \mathbf{u}\|_{L^2} \to 0$$

Discrete compactness property

Discrete analogue for the spaces $V_k \subset \mathbf{H}_0(\mathbf{curl})$ and $Q_k \subset H_0^1$.

For any sequence $\{\mathbf{u}_k\} \subset V_k$ *discretely divergence free*, i.e.,

$$(\mathbf{u}_k, \mathbf{grad}\,\phi_k) = 0 \quad \forall \phi_k \in Q_k,\ \forall k$$

If $\{\mathbf{u}_k\}$ is uniformly bounded in $\mathbf{H}_0(\mathbf{curl})$, $\|\,\mathbf{curl}\,\mathbf{u}_k\|_{L^2} \le 1$, then there exits a subsequence (still denoted $\{\mathbf{u}_k\}$) and $\mathbf{u} \in L^2$ such that

$$\|\mathbf{u}_k - \mathbf{u}\|_{L^2} \to 0$$

Strong DCP

We say that the SDCP is satisfied if $\mathbf{u}$ is divergence free $\operatorname{div}\mathbf{u} = 0$. This is true, for instance, if Q_k is a good approximation to H_0^1.

Commuting diagram property (de Rham complex)

⟨Douglas–Roberts ’82⟩
⟨Bossavit ’88⟩
⟨Arnold ’02⟩
⟨Arnold–Falk–Winther ’10⟩

$Q \subset H_0^1,\ V \subset \mathbf{H}_0(\mathbf{curl}),\ U \subset \mathbf{H}_0(\mathrm{div}),\ S \subset L^2/\mathbb{R}$

$$\begin{array}{ccccccccc} 0 \to & Q & \xrightarrow{\mathbf{grad}} & V & \xrightarrow{\mathbf{curl}} & U & \xrightarrow{\mathrm{div}} & S & \to 0 \\ & \big\downarrow \Pi_k^Q & & \big\downarrow \Pi_k^V & & \big\downarrow \Pi_k^U & & \big\downarrow \Pi_k^S & \\ 0 \to & Q_k & \xrightarrow{\mathbf{grad}} & V_k & \xrightarrow{\mathbf{curl}} & U_k & \xrightarrow{\mathrm{div}} & S_k & \to 0 \end{array}$$

- Kikuchi formulation uses Q and V
- Alternative formulation uses V and U
- U and S are used for Darcy flow or mixed Laplacian

Equivalence

⟨B. '07⟩

Given $V_k \subset \mathbf{H}_0(\mathbf{curl})$, construct Q_k and Z_k such that
$\mathbf{grad}\, Q_k \subset V_k, \quad \mathbf{curl}\, V_k \subset Z_k$

- $Z_k = \mathbf{curl}\, V_k$
- The kernel of **curl** in V_k consists of gradient. Take Q_k as set of potentials vanishing on the boundary $\partial\Omega$

Theorem

The following three sets of conditions are equivalent

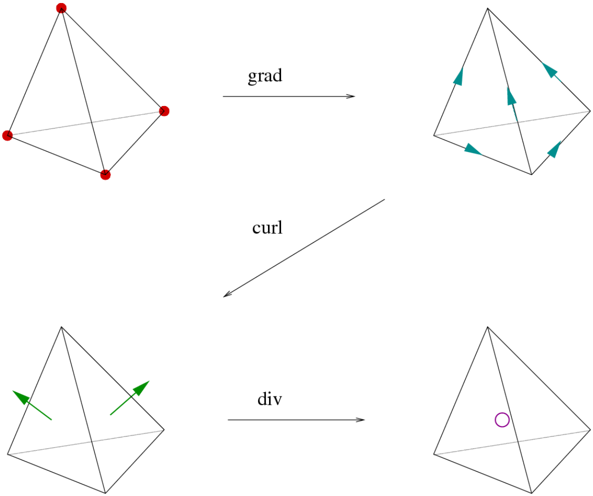
i) *ELKER, WA1, SA1*

ii) *WA2, SA2, FORTID*

iii) *SDCP and standard approximation property: for any* $\mathbf{v} \in V_0$ *there exists* $\mathbf{v}_k^I \in V_k$ *such that*

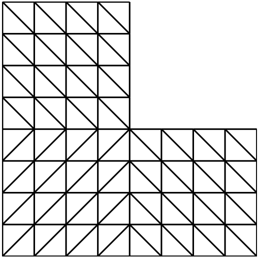
$$\|\mathbf{v} - \mathbf{v}_k^I\|_{curl} \to 0$$

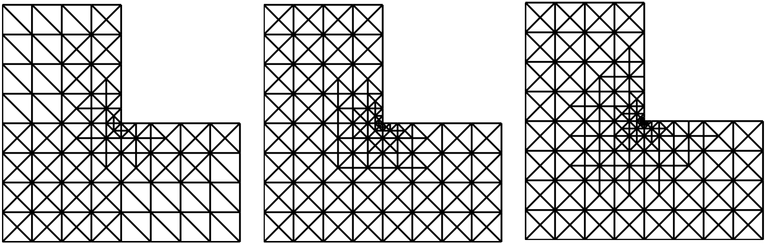
Lowest order finite elements



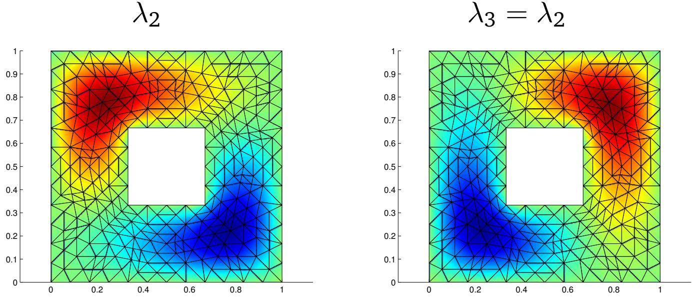
Some comments on adaptive schemes

- A posteriori error analysis
- Convergence study for adaptive schemes





Multiple eigenvalues: the square ring



Question

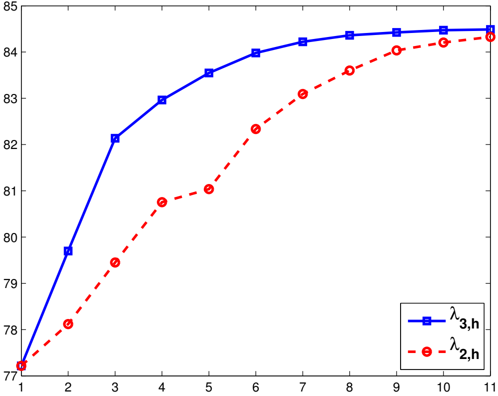
What is the best adaptive strategy for the approximation of the multiple eigenvalue?

1. Indicator based on $(\lambda_{h,2}, u_{h,2})$
2. Indicator based on $(\lambda_{h,3}, u_{h,3})$
3. Indicator based on both $(\lambda_{h,2}, u_{h,2})$ and $(\lambda_{h,3}, u_{h,3})$

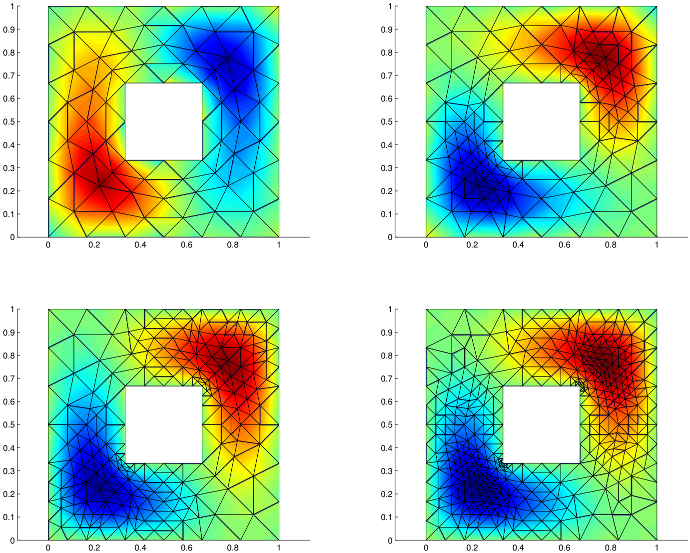
Refinement based on $\lambda_{h,3}$

⟨B.–Durán–Gardini–Gastaldi 2015⟩

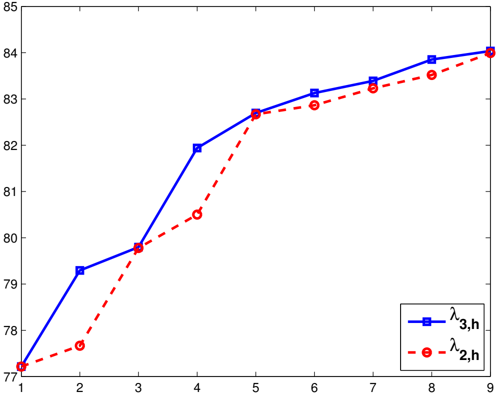
Remark: here we are using a nonconforming discretization which provides eigenvalue approximation from below



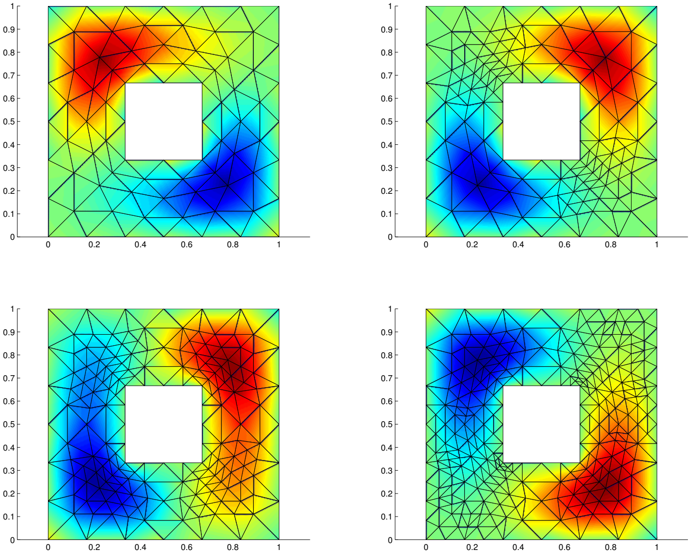
Refinement based on $\lambda_{h,3}$ (eigenfunction $u_{h,3}$)



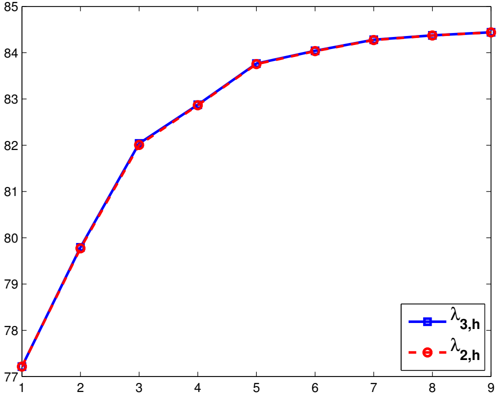
Refinement based on $\lambda_{h,2}$



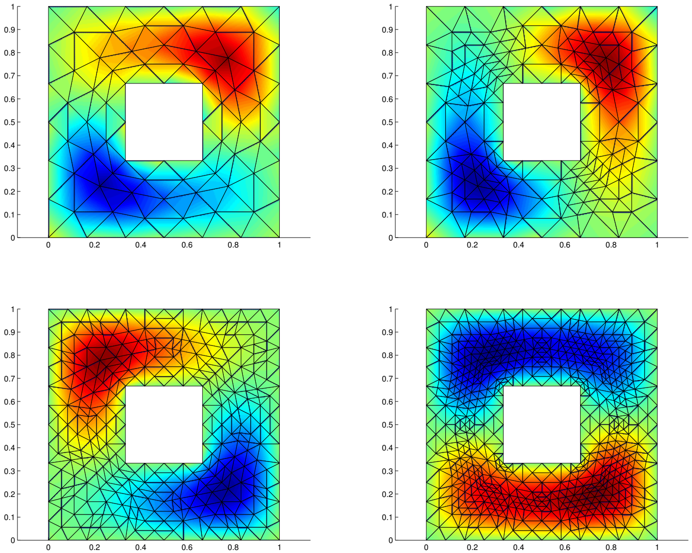
Refinement based on $\lambda_{h,2}$ (eigenfunction $u_{h,2}$)



Refinement based on $\lambda_{h,2}$ and $\lambda_{h,3}$ (eigenvalues)



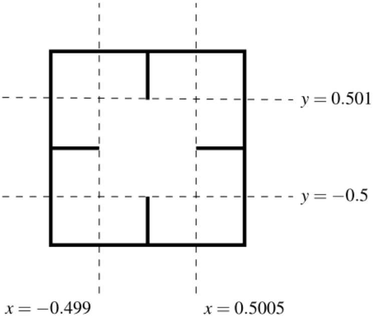
Refinement based on $\lambda_{h,2}$ and $\lambda_{h,3}$ (eigenfunction $u_{h,2}$)

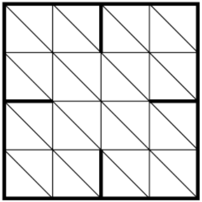


Cluster of eigenvalues

⟨Gallistl ’14⟩

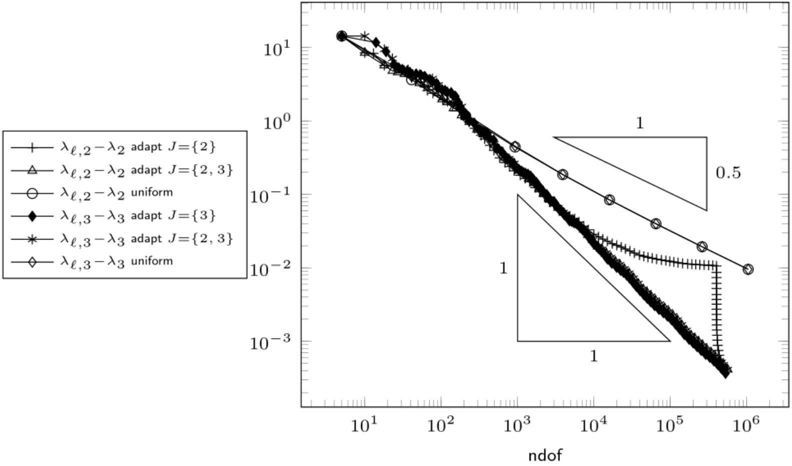
A slightly non-symmetric domain





Now $\lambda_2 < \lambda_3$ but they are very close to each other

Non-symmetric slit domain



From mixed Laplacian to Maxwell's equation

$\mathbf{E}$ electric field

$\left.\begin{array}{l}\varepsilon \text{ electric permittivity} \\ \mu \text{ magnetic permeability}\end{array}\right\} = 1$ (Isotropic and homogeneous material)

$$\begin{cases} \operatorname{curl}(\mu^{-1} \operatorname{curl} \mathbf{E}) = \omega^2 \varepsilon \mathbf{E} & \text{in } \Omega \\ \operatorname{div}(\varepsilon \mathbf{E}) = 0 & \text{in } \Omega \\ \mathbf{E} \times \mathbf{n} = 0 & \text{on } \partial\Omega \end{cases}$$

Mixed formulation ⟨B.–Fernandes–Gastaldi–Perugia '99⟩

Find $\lambda \in \mathbb{R}$ and $(\boldsymbol{\sigma}, \mathbf{p}) \in \boldsymbol{H}_0(\operatorname{curl}) \times \boldsymbol{H}_0(\operatorname{div}^0)$ with $\mathbf{p} \neq 0$ s. t.

$$\begin{cases} (\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\operatorname{curl} \boldsymbol{\tau}, \mathbf{p}) = 0 & \forall \boldsymbol{\tau} \in \boldsymbol{H}_0(\operatorname{curl}) \\ (\operatorname{curl} \boldsymbol{\sigma}, \mathbf{q}) = -\lambda(\mathbf{p}, \mathbf{q}) & \forall \mathbf{q} \in \boldsymbol{H}_0(\operatorname{div}^0) = \operatorname{curl}(\boldsymbol{H}_0(\operatorname{curl})) \end{cases}$$

$\lambda = \omega^2,\ \boldsymbol{\sigma} = \mathbf{E},\ \mathbf{p} = -\operatorname{curl} \mathbf{E}/\lambda$

Approximation of Maxwell's eigenvalue problem

Standard formulation

$\boldsymbol{\mathcal{E}}_h \subset \boldsymbol{H}_0(\text{curl})$ *(edge elements)*

Find $\lambda_h \in \mathbb{R}$ and $\mathbf{E}_h \in \boldsymbol{\mathcal{E}}_h$ with $\mathbf{E}_h \neq 0$ and $\lambda_h \neq 0$ such that

$$(\operatorname{curl} \mathbf{E}_h, \operatorname{curl} \mathbf{F}_h) = \lambda_h (\mathbf{E}_h, \mathbf{F}_h) \quad \forall \mathbf{F}_h \in \boldsymbol{\mathcal{E}}_h$$

Mixed formulation

$\boldsymbol{\mathcal{E}}_h \subset \boldsymbol{H}_0(\text{curl})$ *(edge elements)*

$\boldsymbol{\mathcal{F}}_h = \operatorname{curl} \mathbf{E}_h \subset \boldsymbol{H}_0(\text{div}^0)$ *(face elements)*

Find $\lambda_h \in \mathbb{R}$ and $(\boldsymbol{\sigma}_h, \mathbf{p}_h) \in \boldsymbol{\mathcal{E}}_h \times \boldsymbol{\mathcal{F}}_h$ with $\mathbf{p}_h \neq 0$ such that

$$\begin{cases} (\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + (\operatorname{curl} \boldsymbol{\tau}_h, \mathbf{p}_h) = 0 & \forall \boldsymbol{\tau}_h \in \boldsymbol{\mathcal{E}}_h \\ (\operatorname{curl} \boldsymbol{\sigma}_h, \mathbf{q}_h) = -\lambda_h (\mathbf{p}_h, \mathbf{q}_h) & \forall \mathbf{q}_h \in \boldsymbol{\mathcal{F}}_h \end{cases}$$

Laplace and Maxwell e.p.'s in mixed form

Laplace eigenproblem

Find $\lambda \in \mathbb{R}$ and $(\boldsymbol{\sigma}, u) \in \mathbf{H}_0(\mathrm{div};\Omega) \times L^2_0(\Omega)$ with $u \neq 0$ s. t.

$$\begin{cases} (\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\mathrm{div}\,\boldsymbol{\tau}, u) = 0 & \forall \boldsymbol{\tau} \in \mathbf{H}_0(\mathrm{div};\Omega) \\ (\mathrm{div}\,\boldsymbol{\sigma}, v) = -\lambda(u, v) & \forall v \in L^2_0(\Omega) \end{cases}$$

Maxwell eigenproblem

Find $\lambda \in \mathbb{R}$ and $(\boldsymbol{\sigma}, \mathbf{p}) \in \boldsymbol{H}_0(\mathrm{curl}) \times \boldsymbol{H}_0(\mathrm{div}^0)$ with $\mathbf{p} \neq 0$ s. t.

$$\begin{cases} (\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\mathrm{curl}\,\boldsymbol{\tau}, \mathbf{p}) = 0 & \forall \boldsymbol{\tau} \in \boldsymbol{H}_0(\mathrm{curl}) \\ (\mathrm{curl}\,\boldsymbol{\sigma}, \mathbf{q}) = -\lambda(\mathbf{p}, \mathbf{q}) & \forall \mathbf{q} \in \boldsymbol{H}_0(\mathrm{div}^0) \end{cases}$$

Error indicators

Mixed Laplacian ⟨B.–Gallistl–Gardini–Gastaldi '16⟩

$$\eta_L(K)^2 = \|h_K(\sigma_{h,j} - \nabla u_{h,j})\|^2_{0,K} + \|h_K \operatorname{curl} \sigma_{h,j}\|^2_{0,K} + \frac{1}{2} \sum_{F \in \mathcal{F}(K)} h_F \|[\sigma_{h,j}]_F \times n_F\|^2_F$$

Maxwell in mixed form ⟨B.–Gastaldi–Rodríguez–Šebestová '16⟩

$$\eta_{MM}(K)^2 = \|h_K(\boldsymbol{\sigma}_h + \operatorname{curl} \mathbf{p}_h)\|^2_{0,K} + \|h_K \operatorname{div} \boldsymbol{\sigma}_h\|^2_{0,K} + \frac{1}{2} \sum_{F \in \mathbf{F}(K)} \left(h_F \|[\mathbf{p}_h \times \mathbf{n}]\|^2_{0,F} + h_F \|[\boldsymbol{\sigma}_h \cdot \mathbf{n}]\|^2_{0,F} \right)$$

Standard Maxwell formulation $\boldsymbol{\sigma}_h = \mathbf{E}_h, \ \mathbf{p}_h = -\operatorname{curl} \mathbf{E}_h / \lambda_h$

$$\eta_{SM}(K)^2 = \|h_K(\mathbf{E}_h - \operatorname{curl}(\operatorname{curl} \mathbf{E}_h/\lambda_h)\|^2_{0,K} + \|h_K \operatorname{div} \mathbf{E}_h\|^2_{0,K} + \frac{1}{2} \sum_{F \in \mathbf{F}(K)} \left(h_F \|[(\operatorname{curl} \mathbf{E}_h/\lambda_h) \times \mathbf{n}]\|^2_{0,F} + h_F \|[\mathbf{E}_h \cdot \mathbf{n}]\|^2_{0,F} \right)$$

Convergence analysis (AFEM for mixed Laplace)

Input

Parameter $\theta \in (0,1]$ and initial triangulation $\mathcal{T}_0$

SOLVE, ESTIMATE, MARK, REFINE

Solve: Compute discrete solution $(\lambda_\ell, \sigma_\ell, u_\ell)$ on $\mathcal{T}_\ell$

Estimate: Compute local contributions of the error estimator $\{\eta_\ell^2(T)\}_{T\in\mathcal{T}_\ell}$

Mark: Choose minimal subset $\mathcal{M}_\ell \subset \mathcal{T}_\ell$ such that $\theta\eta_\ell^2(\mathcal{T}_\ell) \le \eta_\ell^2(\mathcal{M}_\ell) \quad (0 < \theta \le 1)$

Refine: Generate new triangulation as the smallest refinement of $\mathcal{T}_\ell$ satisfying $\mathcal{M}_\ell \cap \mathcal{T}_{\ell+1} = \emptyset$

Output

Sequence of meshes $\{\mathcal{T}_\ell\}$, sol.'s $\{(\lambda_\ell, \sigma_\ell, u_\ell)\}$, indicators $\{\eta_\ell(\mathcal{T}_\ell)\}$

Cluster of length $\mathbf{N}$

$\lambda_{n+1}, \ldots, \lambda_{n+\mathbf{N}}$
$J = \{n+1, \ldots, n+\mathbf{N}\}$

Corresponding combination of eigenspaces

$W = \operatorname{span}\{u_j \mid j \in J\}$
$W_{\mathcal{T}_h} = W_h = \operatorname{span}\{u_{h,j} \mid j \in J\}$

How to implement the AFEM scheme

Consider contribution of all elements in W_ℓ simultaneously

$$\theta \sum_{j \in J} \eta_{\ell,j}(\mathcal{T}_\ell)^2 \leq \sum_{j \in J} \eta_{\ell,j}(\mathcal{M}_\ell)^2$$

Error quantity

Let us introduce the gradient $\mathbf{G}$ and the discrete gradient $\mathbf{G}_h$

$\mathbf{G}(w) \in H(\mathrm{div};\Omega)$ is the solution to

$$(\mathbf{G}(w),\tau) + (\mathrm{div}\,\tau, w) = 0 \quad \text{for all } \tau \in H(\mathrm{div};\Omega)$$

$\mathbf{G}_h(w_h) \in \Sigma_h$ is the solution to

$$(\mathbf{G}_h(w_h),\tau_h) + (\mathrm{div}\,\tau_h, w_h) = 0 \quad \text{for all } \tau_h \in \Sigma_h.$$

Error quantity

$$d(v,w) = \sqrt{\|v-w\|^2 + \|\mathbf{G}(v) - \mathbf{G}(w)\|^2}$$

N.B: when v (resp. w) belongs to M_h, then $\mathbf{G}_h(v)$ (resp. $\mathbf{G}_h(w)$) should be used

$$\boxed{\delta(W,W_h) = \sup_{\substack{u\in W\\ \|u\|=1}} \inf_{v_h\in W_h} d(u,v_h)}$$

Main theorem (convergence and optimal rate)

Nonlinear approximation classes ⟨Binev–Dahmen–DeVore 2004⟩
⟨Stevenson 2007⟩
⟨Cascon–Kreuzer–Nochetto–Siebert 2008⟩

Best convergence rate $s \in (0, +\infty)$ characterized in terms of

$$|W|_{\mathcal{A}_s} = \sup_{m \in \mathbb{N}} m^s \inf_{\mathcal{T} \in \mathbb{T}(m)} \delta(W, W_{\mathcal{T}}).$$

In particular, $|W|_{\mathcal{A}_s} < \infty$ if $\delta(W, W_{\mathcal{T}}) = O(m^{-s})$ for the optimal triangulations in $\mathbb{T}(m)$, that is, with $\mathrm{card}(\mathcal{T}) - \mathrm{card}(\mathcal{T}_0) \leq m$

Theorem (B.–Gallistl–Gardini–Gastaldi ’16)

Provided the initial mesh-size and the bulk parameter θ are small enough, if for the eigenvalue cluster W it holds $|W|_{\mathcal{A}_s} < \infty$, then the sequence of discrete clusters W_ℓ computed on the mesh $\mathcal{T}_\ell$ satisfies the optimal estimate

$$\delta(W, W_\ell)(\mathrm{card}(\mathcal{T}_\ell) - \mathrm{card}(\mathcal{T}_0))^s \leq C|W|_{\mathcal{A}_s}$$

Convergence of the eigenvalues

The previous theorem implies that the eigenfunctions in the cluster are optimally approximated. The next theorem shows that the eigenvalues are well approximated as well

Theorem (B.–Gallistl–Gardini–Gastaldi ’16)

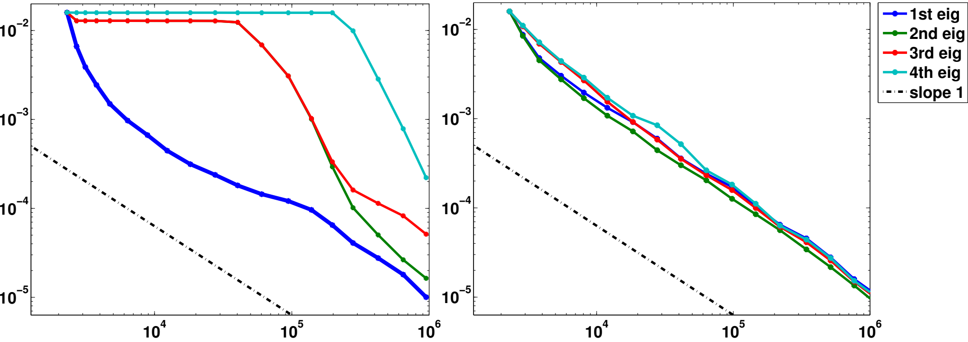
Let J denote the set of indices corresponding to the eigenvalues in the cluster W. Then

$$\sup_{i\in J}\inf_{j\in J}|\lambda_i-\lambda_{\ell,j}|\le C\delta(W,W_\ell)^2$$

Conclusions

- Standard Galerkin formulation: methods working for the source problem can be successfully applied to corresponding eigenvalue problem (pointwise convergence implies uniform convergence thanks to compactness)
- Mixed formulations: approximation of eigenvalue problems require different conditions than corresponding source problems (pointwise vs. uniform convergence)
- Multiple eigenvalues and clusters of eigenvalues need particular attention (a priori and a posteriori)
- A posteriori analysis: need for a new paradigm?

Convergence rate vs. computational cost



Influence of the bulk parameter