Finite Element Exterior Calculus

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- The Hilbert complex framework
- Discretization of Hilbert complexes
- Finite element differential forms
- FEDF on cubical meshes
- New complexes from old

The Hilbert complex framework

Short closed Hilbert complex

$$W^0 \xrightarrow{(d^0, V^0)} W^1 \xrightarrow{(d^1, V^1)} W^2$$

- d^i are closed, densely-defined w/ closed range, $d^1 \circ d^0 = 0$
- Null space & range: $\mathfrak{Z}^i := \mathcal{N}(d^i), \mathfrak{B}^i := \mathcal{R}(d^{i-1}), \mathfrak{B}^1 \subseteq \mathfrak{Z}^1$
- Dual complex: $W^0 \xleftarrow{(d_1^*, V_1^*)} W^1 \xleftarrow{(d_2^*, V_2^*)} W^2$
- Harmonic forms: $\mathfrak{H} = \mathfrak{Z}^1 \cap \mathfrak{Z}_1^* = (\mathfrak{B}^1)^{\perp_{\mathfrak{Z}^1}} \cong \mathfrak{B}^1 / \mathfrak{Z}^1$ (homology)
- Hodge decomposition: $W^1 = \underbrace{\overbrace{\mathfrak{B}^1 \oplus \mathfrak{H}}^{\mathfrak{Z}}}_{\mathfrak{Z}^{*\perp}} \oplus \overbrace{\mathfrak{B}_1^*}^{\mathfrak{Z}^\perp}$, with $\underbrace{\mathfrak{B}^1}_{\mathfrak{Z}^{*\perp}}$ and $\underbrace{\mathfrak{H} \oplus \mathfrak{B}_1^*}_{\mathfrak{Z}^*}$
- Poincaré inequality: $\|u\| \le C_P \|du\|, \quad u \in V \cap \mathfrak{Z}^\perp$

Hodge Laplacian

$$W^0 \underset{d_1^*}{\overset{d^0}{\rightleftarrows}} W^1 \underset{d_2^*}{\overset{d^1}{\rightleftarrows}} W^2$$

- $L = dd^* + d^*d : W^1 \to W^1$,
  $D(L) = \{u \in V \cap V^* \,|\, du \in V^*,\ d^*u \in V\}$
- $L = L^*$, $\mathcal{N}(L) = \mathcal{R}(L)^\perp = \mathfrak{H}$
- *Strong form:* Given $f \in W$ find $u \in D(L)$ s.t. $Lu = f \pmod{\mathfrak{H}}$, $u \perp \mathfrak{H}$
- *Primal weak form:* Find $u \in V \cap V^* \cap \mathfrak{H}^\perp$ s.t.

$$\langle du, dv\rangle + \langle d^*u, d^*v\rangle = \langle f, v\rangle, \quad v \in V \cap V^* \cap \mathfrak{H}^\perp$$

- *Mixed weak form:* Find $\sigma \in V^0, u \in V^1, p \in \mathfrak{H}$ s.t.

$$\begin{aligned} \langle \sigma, \tau\rangle - \langle u, d\tau\rangle &= 0, && \tau \in V^0, \\ \langle d\sigma, v\rangle + \langle du, dv\rangle + \langle p, v\rangle &= \langle f, v\rangle, && v \in V^1, \\ \langle u, q\rangle &= 0, && q \in \mathfrak{H}. \end{aligned}$$

Well-posedness

THEOREM

Let $W^0 \xrightarrow{d} W^1 \xrightarrow{d} W^2$ *be a closed H-complex. Then*

1. $\forall f \in W^1\ \exists!\ (\sigma, u, p) \in V^0 \times V^1 \times \mathfrak{H}$ *solving the weak mixed form.*
2. $\exists\, c$ *depending only on the Poincaré constant for the complex s.t.*
$$\|\sigma\|_V + \|u\|_V + \|p\| \le c\|f\|$$
3. $u \in D(L)$ *and satisfies* $Lu = f \pmod{\mathfrak{H}}$, $u \perp \mathfrak{H}$. *(weak* $\Longleftrightarrow$ *strong)*

To show: inf-sup cond'n for $\langle\sigma,\tau\rangle - \langle u, d\tau\rangle + \langle d\sigma, v\rangle + \langle du, dv\rangle + \langle p, v\rangle + \langle u, q\rangle$.
So we need to bound $\|\sigma\|_V + \|u\|_V + \|p\|$ by bounded choice of τ, v, q.

Easy to bound $\|\sigma\|, \|d\sigma\|, \|p\|, \|du\|$, but … *what about* $\|u\|$*?*

Hodge decomp of W^1*:* $u = u_{\mathfrak{B}} + u_{\mathfrak{H}} + u_{\mathfrak{B}^*}$.

Easy to bound $\|u_{\mathfrak{H}}\|$. To bound $u_{\mathfrak{B}}$ take $\tau \in V^0 \cap \mathfrak{Z}^\perp$ with $d\tau = u_{\mathfrak{B}}$ and use *Poincaré's inequality in* V^0.

Finally $\|u_{\mathfrak{B}^*}\| \le C_P\|du_{\mathfrak{B}^*}\| = C_P\|du\|$ by *Poincaré's inequality in* V^1.

Examples

Ex 0. $\Omega \subset \mathbb{R}^3$ $\qquad 0 \longrightarrow L^2(\Omega) \xrightarrow{(\mathrm{grad}, H^1)} L^2(\Omega) \otimes \mathbb{R}^3$

Mixed=Primal: Find $u \in H^1,\ p \in \mathbb{R}$ s.t.

$$\langle \mathrm{grad}\, u, \mathrm{grad}\, v\rangle = \langle f - p, v\rangle, \quad v \in H^1, \qquad \int u = 0.$$

std. Poisson
Neumann BC

Examples

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std. Poisson
Neumann BC

Ex 0a. $$0 \longrightarrow L^2(\Omega) \xrightarrow{(\mathrm{grad}, \mathring{H}^1)} L^2(\Omega) \otimes \mathbb{R}^3$$

Find $u \in \mathring{H}^1$ s.t. $\langle \mathrm{grad}\, u, \mathrm{grad}\, v\rangle = \langle f, v\rangle, \quad v \in \mathring{H}^1.$

Examples

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Find $u \in \mathring{H}^1$ s.t. $\langle \mathrm{grad}\, u, \mathrm{grad}\, v \rangle = \langle f, v \rangle, \quad v \in \mathring{H}^1.$

Ex 3. $\quad L^2(\Omega) \otimes \mathbb{R}^3 \xrightarrow{(\mathrm{div}, H(\mathrm{div}))} L^2(\Omega) \longrightarrow 0$

Mixed: Find $\sigma \in H(\mathrm{div})$, $u \in L^2$ s.t.

$$\langle \sigma, \tau \rangle - \langle u, \mathrm{div}\, \tau \rangle = 0, \qquad \tau \in H(\mathrm{div}),$$

$$\langle \mathrm{div}\, \sigma, v \rangle = \langle f, v \rangle, \qquad v \in L^2.$$

mixed Poisson

The de Rham complex in 3D

$$0 \to L^2 \xrightarrow{\mathrm{grad}, H^1} L^2 \otimes \mathbb{R}^3 \xrightarrow{\mathrm{curl}, H(\mathrm{curl})} L^2 \otimes \mathbb{R}^3 \xrightarrow{\mathrm{div}, H(\mathrm{div})} L^2 \to 0$$

The de Rham complex in 3D

$$0 \to L^2 \underset{-\operatorname{div}, \mathring{H}(\operatorname{div})}{\overset{\operatorname{grad}, H^1}{\rightleftarrows}} L^2 \otimes \mathbb{R}^3 \underset{\operatorname{curl}, \mathring{H}(\operatorname{curl})}{\overset{\operatorname{curl}, H(\operatorname{curl})}{\rightleftarrows}} L^2 \otimes \mathbb{R}^3 \underset{-\operatorname{grad}, \mathring{H}^1}{\overset{\operatorname{div}, H(\operatorname{div})}{\rightleftarrows}} L^2 \to 0$$

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$$0 \to L^2 \underset{-\operatorname{div}, \mathring{H}(\operatorname{div})}{\overset{\operatorname{grad}, H^1}{\rightleftarrows}} L^2 \otimes \mathbb{R}^3 \underset{\operatorname{curl}, \mathring{H}(\operatorname{curl})}{\overset{\operatorname{curl}, H(\operatorname{curl})}{\rightleftarrows}} L^2 \otimes \mathbb{R}^3 \underset{-\operatorname{grad}, \mathring{H}^1}{\overset{\operatorname{div}, H(\operatorname{div})}{\rightleftarrows}} L^2 \to 0$$

Ex 1. $dd^* + d^*d = -\operatorname{grad}\operatorname{div} + \operatorname{curl}\operatorname{curl}$

Natural magnetic b.c.: $u \cdot n = 0, \quad \operatorname{curl} u \times n = 0$

$\sigma = -\operatorname{div} u$

Ex 2. $dd^* + d^*d = \operatorname{curl}\operatorname{curl} - \operatorname{grad}\operatorname{div}$

Natural electric b.c.: $u \times n = 0, \quad \operatorname{div} u = 0$

$\sigma = \operatorname{curl} u$

The dimensions of the homology groups (harmonic forms) are the Betti numbers of Ω.

The Hodge wave equation

$$\ddot{U} + (dd^* + d^*d)U = 0, \quad U(0) = U_0, \quad \dot{U}(0) = U_1$$

Then $\sigma := d^*U,\ \rho := dU,\ u := \dot{U}$ satisfy

$$\begin{pmatrix} \dot\sigma \\ \dot u \\ \dot\rho \end{pmatrix} + \begin{pmatrix} 0 & -d^* & 0 \\ d & 0 & d^* \\ 0 & -d & 0 \end{pmatrix} \begin{pmatrix} \sigma \\ u \\ \rho \end{pmatrix} = 0$$

strong

Find $\quad (\sigma, u, \rho) : [0, T] \to V^0 \times V^1 \times W^2 \quad$ s.t.

$$\begin{aligned} \langle \dot\sigma, \tau \rangle \ - \ \langle u, d\tau \rangle &= 0, && \tau \in V^0, \\ \langle \dot u, v \rangle \ + \ \langle d\sigma, v \rangle + \langle \rho, dv \rangle &= 0, && v \in V^1, \\ \langle \dot\rho, \eta \rangle \ - \ \langle du, \eta \rangle &= 0, && \eta \in W^2. \end{aligned}$$

weak

Theorem

Given initial data $(\sigma_0, u_0, \rho_0) \in V^0 \times V^1 \times W^2$, $\exists!$ *solution* $(\sigma, u, \rho) \in C^0([0,T], V^0 \times V^1 \times W^2) \cap C^1([0,T], W^0 \times W^1 \times W^2)$.

Proof: Uniqueness: $(\tau, v, \eta) = (\sigma, u, \rho)$. Existence: Hille–Yosida.

Example: Maxwell's equations

$$\dot{D} = \operatorname{curl} H \qquad \dot{B} = -\operatorname{curl} E$$
$$\operatorname{div} D = 0 \qquad \operatorname{div} B = 0$$
$$D = \epsilon E \qquad B = \mu H$$

$$W^0 = L^2(\Omega)$$
$$W^1 = L^2(\Omega, \mathbb{R}^3, \epsilon\, dx)$$
$$W^2 = L^2(\Omega, \mathbb{R}^3, \mu^{-1} dx)$$

$$W^0 \xrightarrow{\operatorname{grad}} W^1 \xrightarrow{-\operatorname{curl}} W^2$$

$(\sigma, E, B) : [0, T] \times \Omega \to \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$ solves

$$\langle \dot{\sigma}, \tau \rangle - \langle \epsilon E, \operatorname{grad} \tau \rangle = 0 \ \forall \tau,$$
$$\langle \epsilon \dot{E}, F \rangle + \langle \epsilon \operatorname{grad} \sigma, F \rangle - \langle \mu^{-1} B, \operatorname{curl} F \rangle = 0 \ \forall F,$$
$$\langle \mu^{-1} \dot{B}, C \rangle + \langle \mu^{-1} \operatorname{curl} E, C \rangle = 0 \ \forall C.$$

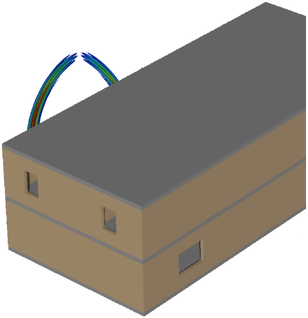
Theorem

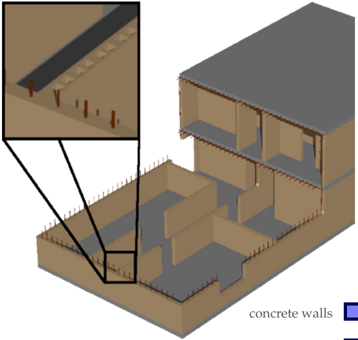
If σ, $\operatorname{div} \epsilon E$, and $\operatorname{div} B$ vanish for $t = 0$, then they vanish for all t, and E, B, $D = \epsilon E$, and $H = \mu^{-1} B$ satisfy Maxwell's equations.

Simulation of radar scattering off building

Stowell–Fassenfass–White, IEEE Trans. Ant. & Prop. 2008

- Solved time-dependent Maxwell equations using $\mathcal{Q}_1^-\Lambda^1$ for E and $\mathcal{Q}_1^-\Lambda^2$ for B (Nédélec elements of the first kind on bricks)
- 10,114,695,855 brick elements ($\approx$ 1 cm resolution)
- $\approx 60,000,000,000$ unknowns
- $\approx 12,000$ time steps of 14 picoseconds





concrete walls

cinder block/rebar walls

Some other complexes

$$0 \to \underset{\text{displacement}}{L^2 \otimes \mathbb{R}^3} \xrightarrow{\text{sym grad}} \underset{\text{strain}}{L^2 \otimes \mathbb{S}^3} \xrightarrow{\text{curl T curl}} \underset{\text{stress}}{L^2 \otimes \mathbb{S}^3} \xrightarrow{\text{div}} \underset{\text{load}}{L^2 \otimes \mathbb{R}^3} \to 0$$

- $0 \to L^2 \otimes \mathbb{R}^3 \xrightarrow{\text{sym grad}} L^2 \otimes \mathbb{S}^3$ primal method for elasticity
- $L^2 \otimes \mathbb{S}^3 \xrightarrow{\text{div}} L^2 \otimes \mathbb{R}^3 \to 0$ mixed method for elasticity

Some other complexes

$$0 \to L^2 \otimes \mathbb{R}^3 \xrightarrow{\text{sym grad}} L^2 \otimes \mathbb{S}^3 \xrightarrow{\text{curl T curl}} L^2 \otimes \mathbb{S}^3 \xrightarrow{\text{div}} L^2 \otimes \mathbb{R}^3 \to 0$$

displacement   strain   stress   load

- $0 \to L^2 \otimes \mathbb{R}^3 \xrightarrow{\text{sym grad}} L^2 \otimes \mathbb{S}^3$ primal method for elasticity
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$\mathbb{R}^{3\times 3}$ trace-free

$$0 \to L^2 \xrightarrow{\text{grad grad}} L^2 \otimes \mathbb{S}^3 \xrightarrow{\text{curl}} L^2 \otimes \mathbb{T} \xrightarrow{\text{div}} L^2 \otimes \mathbb{R}^3 \to 0$$

- $0 \to L^2 \xrightarrow{\text{grad grad}} L^2 \otimes \mathbb{S}^3$ primal method for plate equation
- $L^2 \xrightarrow{\text{grad grad}} L^2 \otimes \mathbb{S}^3 \xrightarrow{\text{curl}} L^2 \otimes \mathbb{V}$ Einstein–Bianchi eqs (GR)

Discretization of Hilbert complexes

Discretization

Choose finite dimensional spaces $V_h^0 \subset V^0$, $V_h^1 \subset V^1$.
Of course they must have reasonable approximation:

$$\lim_{h \to 0} \inf_{v \in V_h} \|u - v\|_W = 0, \quad u \in W.$$

Let $d_h = d|_{V_h}$, $\mathfrak{Z}_h = \mathcal{N}(d_h)$, $\mathfrak{B}_h = \mathcal{R}(d_h)$, $\mathfrak{H}_h = \mathfrak{Z}_h \cap \mathfrak{B}_h^\perp$

Galerkin's method: Find $(\sigma_h, u_h, p_h) \in V_h^0 \times V_h^1 \times \mathfrak{H}^h$ s.t.

$$\begin{aligned}
\langle \sigma_h, \tau \rangle - \langle u_h, d\tau \rangle &= 0, & \tau &\in V_h^0, \\
\langle d\sigma_h, v \rangle + \langle du_h, dv \rangle + \langle p_h, v \rangle &= \langle f, v \rangle, & v &\in V_h^1, \\
\langle u_h, q \rangle &= 0, & q &\in \mathfrak{H}_h.
\end{aligned}$$

When is this approximation stable, consistent, and convergent?

Assumptions on the discretization

Besides approximation, there are two key requirements:

Subcomplex assumption: $d\,V_h^k \subset V_h^{k+1}$

Bounded Cochain Projection assumption: $\exists\, \pi_h^k : V^k \to V_h^k$

$$\begin{tikzcd} V^k \arrow[r, "d^k"] \arrow[d, "\pi_h^k"] & V^{k+1} \arrow[d, "\pi_h^{k+1}"] \\ V_h^k \arrow[r, "d^k"] & V_h^{k+1} \end{tikzcd}$$

- π_h^k is bounded, uniformly in h
- π_h^k preserves V_h^k
- $\pi_h^{k+1} d^k = d^k \pi_h^k$

Consequences of the assumptions

The subcomplex property implies that $V_h^0 \xrightarrow{d_h} V_h^1 \xrightarrow{d_h} W^2$ is itself an H-complex. So it has its own harmonic forms, Hodge decomposition, and Poincaré inequality. The Hodge Laplacian for the discrete complex is the Galerkin method for the original problem.

Theorem

Given the approximation, subcomplex, and BCP assumptions:

- $\mathfrak{H} \cong \mathfrak{H}_h$ *and* $\operatorname{gap}(\mathfrak{H}, \mathfrak{H}_h) \to 0$.
- *The Galerkin method is consistent.*
- *The discrete Poincaré inequality* $\|\omega\| \le c\|d\omega\|, \quad \omega \in \mathfrak{Z}_h^{k\perp}$, *holds with* c *independent of* h.
- *The Galerkin method is stable.*
- *The Galerkin method is convergent with the error estimate:*

$$\|\sigma - \sigma_h\|_V + \|u - u_h\|_V + \|p - p_h\|_V \le c[\, E(\sigma) + E(u) + E(p) + \epsilon\,]$$

where $\epsilon \le E(P_{\mathfrak{B}} u) \cdot \sup_{r \in \mathfrak{H}^k, \|r\|=1} E(r)$.

Finite element differential forms

Differential forms on a domain $\Omega \subset \mathbb{R}^n$

- Differential k-forms are functions $\Omega \to \mathrm{Alt}^k\mathbb{R}^n$

  0-forms: functions; 1-forms: covector fields; k-forms: $\binom{n}{k}$ components

  $$u = \sum_\sigma f_\sigma \, dx^\sigma := \sum_{1\le\sigma_1<\cdots<\sigma_k\le n} f_{\sigma_1\cdots\sigma_k} \, dx^{\sigma_1} \wedge \cdots \wedge dx^{\sigma_k}$$

- The wedge product of a k-form and an l-form is a $(k+l)$-form
- The exterior derivative du of a k-form is a $(k+1)$-form
- A k-form can be integrated over a k-dimensional subset of Ω
- Given $F : \Omega \to \Omega'$, a k-form on Ω' can be pulled back to a k-form on Ω.
- The trace of a k-form on a submanifold is the pull back under inclusion.
- Stokes theorem: $\displaystyle\int_\Omega du = \int_{\partial\Omega} \operatorname{tr} u, \quad u \in \Lambda^{k-1}(\Omega)$
- The exterior derivative can be viewed as a closed, densely-defined op $L^2\Lambda^k \to L^2\Lambda^{k+1}$ with domain $H\Lambda^k(\Omega) = \{\, u \in L^2\Lambda^k \mid du \in L^2\Lambda^{k+1} \,\}$. If Ω has a Lipschitz boundary, it has closed range.

The L^2 de Rham complex and its discretization

$$0 \to L^2\Lambda^0 \xrightarrow{d} L^2\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} L^2\Lambda^n \to 0$$

Our goal is to define spaces $V_h^k \subset H\Lambda^k$ satisfying the approximation, subcomplex, and BCP assumptions.

In the case $k = 0$, $V_h^k \subset H^1$ will just be the Lagrange elements. It turns out that for $k > 0$ there are two distinct generalizations.

The Lagrange finite element space $\mathcal{P}_r\Lambda^0(\mathcal{T}_h)$

nothing new
(but notation)

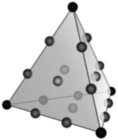
Elements: A triangulation $\mathcal{T}_h$ consisting of simplices T

Shape functions: $V(T) = \mathcal{P}_r\Lambda^0(T) = \mathcal{P}_r(T)$, some $r \geq 1$

Degrees of freedom (*unisolvent*):

- $v \in \Delta_0(T)$: $u \mapsto u(v)$
- $e \in \Delta_1(T)$: $u \mapsto \int_e (\mathrm{tr}_e\, u) q\, dt$, $q \in \mathcal{P}_{r-2}(e)$
- $f \in \Delta_2(T)$: $u \mapsto \int_f (\mathrm{tr}_f\, u) q\, ds$, $q \in \mathcal{P}_{r-3}(f)$

$\vdots$



$$u \mapsto \int_f (\mathrm{tr}_f\, u) \wedge q, \quad q \in \mathcal{P}_{r-d-1}\Lambda^d(f),\ f \in \Delta_d(T),\ d \geq 0$$

The resulting finite element space belongs to $H\Lambda^0 = H^1$. In fact, it equals

$$\{\, u \in H\Lambda^0(\Omega) \;:\; u|_T \in V(T)\ \forall T \in \mathcal{T}_h \,\}$$

The $\mathcal{P}_r\Lambda^k(\mathcal{T}_h)$ and $\mathcal{P}_r^-\Lambda^k(\mathcal{T}_h)$ finite element spaces

Elements: A triangulation $\mathcal{T}_h$ consisting of simplices T

to be defined

Shape functions: $V(T) = \mathcal{P}_r\Lambda^k(T)$ or $\mathcal{P}_r^-\Lambda^k(T)$, some $r \geq 1$

Degrees of freedom for $\mathcal{P}_r\Lambda^k$ (*unisolvent*):

$$u \mapsto \int_f (\operatorname{tr}_f u) \wedge q, \quad q \in \mathcal{P}_{r-d+k}^-\Lambda^{d-k}(f),\ f \in \Delta_d(T),\ d \geq k$$

Degrees of freedom for $\mathcal{P}_r^-\Lambda^k$ (*unisolvent*):

$$u \mapsto \int_f (\operatorname{tr}_f u) \wedge q, \quad q \in \mathcal{P}_{r-d+k-1}\Lambda^{d-k}(f),\ f \in \Delta_d(T),\ d \geq k$$

Hiptmair '99

The resulting finite element spaces belongs to $H\Lambda^k$. In fact, they equal

$$\{\, u \in H\Lambda^0(\Omega) \ :\ u|_T \in V(T)\ \forall T \in \mathcal{T}_h \,\}$$

The Koszul differential

κ is a simple algebraic operation take k-forms to $k-1$-forms.
It takes polynomial forms to polynomial forms of 1 degree higher.

- $\kappa(dx^i) = x^i, \quad \kappa(u \wedge v) = (\kappa u) \wedge v + (-)^k u \wedge (\kappa v)$
- $\kappa(f\, dx^{\sigma_1} \wedge \cdots \wedge dx^{\sigma_k}) = \sum_{i=1}^k (-)^i f\, x^{\sigma_i}\, dx^{\sigma_1} \wedge \cdots \widehat{dx^{\sigma_i}} \cdots \wedge dx^{\sigma_k}$
- In $\mathbb{R}^3$: $\quad \mathcal{P}_r\Lambda^3 \xrightarrow{\vec{x}} \mathcal{P}_{r+1}\Lambda^2 \xrightarrow{\times\vec{x}} \mathcal{P}_{r+2}\Lambda^1 \xrightarrow{\cdot\vec{x}} \mathcal{P}_{r+3}\Lambda^0$

Two basic properties:

- κ is a *differential:* $\quad \kappa \circ \kappa = 0$
- *Homotopy property:* $\quad (d\kappa + \kappa d)u = (r+k)u, \; u \in \mathcal{H}_r\Lambda^k$

homog. deg. r

e.g., $\quad \operatorname{curl}(\vec{x} \times \vec{v}) + \vec{x}\,(\operatorname{div} \vec{v}) = (\deg \vec{v} + 2)\,\vec{v}$

Consequences:

- The *Koszul complex* $0 \to \mathcal{H}_r\Lambda^n \xrightarrow{\kappa} \mathcal{H}_{r+1}\Lambda^{n-1} \xrightarrow{\kappa} \cdots \xrightarrow{\kappa} \mathcal{H}_{r+n}\Lambda^0 \to 0$ is exact.
- $\mathcal{H}_r\Lambda^k = \kappa\, \mathcal{H}_{r-1}\Lambda^{k+1} \oplus d\, \mathcal{H}_{r+1}\Lambda^{k-1}$

Definition: $\quad \mathcal{P}_r^-\Lambda^k = \mathcal{P}_{r-1}\Lambda^k + \kappa\mathcal{H}_{r-1}\Lambda^{k+1}$

Finite element discretizations of the de Rham complex

- $\mathcal{P}_{r-1}\Lambda^k \subsetneq \mathcal{P}_r^-\Lambda^k \subsetneq \mathcal{P}_r\Lambda^k$ except $\mathcal{P}_r^-\Lambda^0 = \mathcal{P}_r\Lambda^0$, $\mathcal{P}_r^-\Lambda^n = \mathcal{P}_{r-1}\Lambda^n$)
- The DOFs given above for $\mathcal{P}_r\Lambda^k$ and $\mathcal{P}_r^-\Lambda^k$ are *unisolvent*.
- The DOFs define cochain projections for the complexes

$$0 \to \mathcal{P}_r\Lambda^0(\mathcal{T}_h) \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^1(\mathcal{T}_h) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_{r-n}\Lambda^n(\mathcal{T}_h) \to 0$$

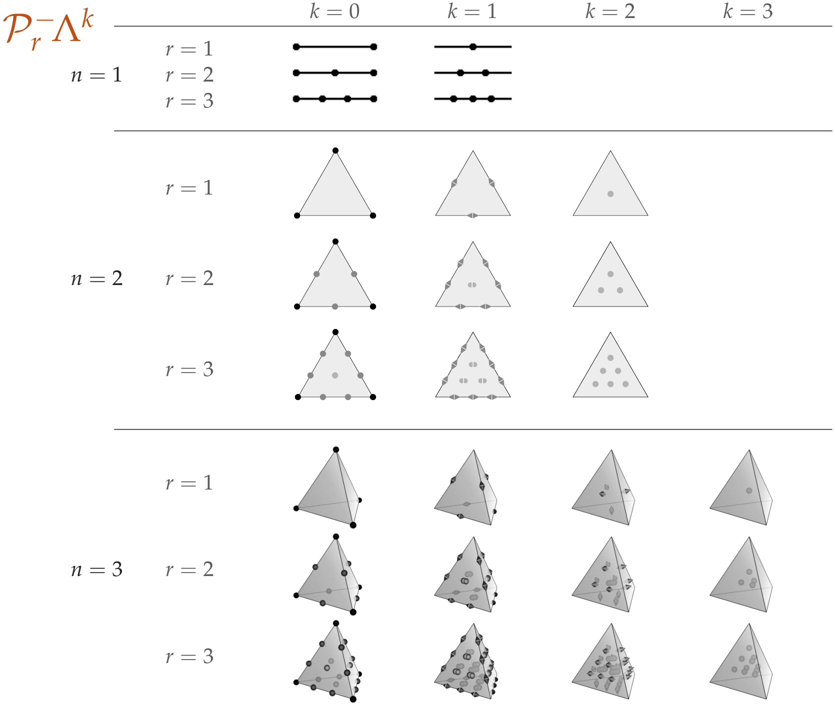
$$0 \to \mathcal{P}_r^-\Lambda^0(\mathcal{T}_h) \xrightarrow{d} \mathcal{P}_r^-\Lambda^1(\mathcal{T}_h) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_r^-\Lambda^n(\mathcal{T}_h) \to 0$$

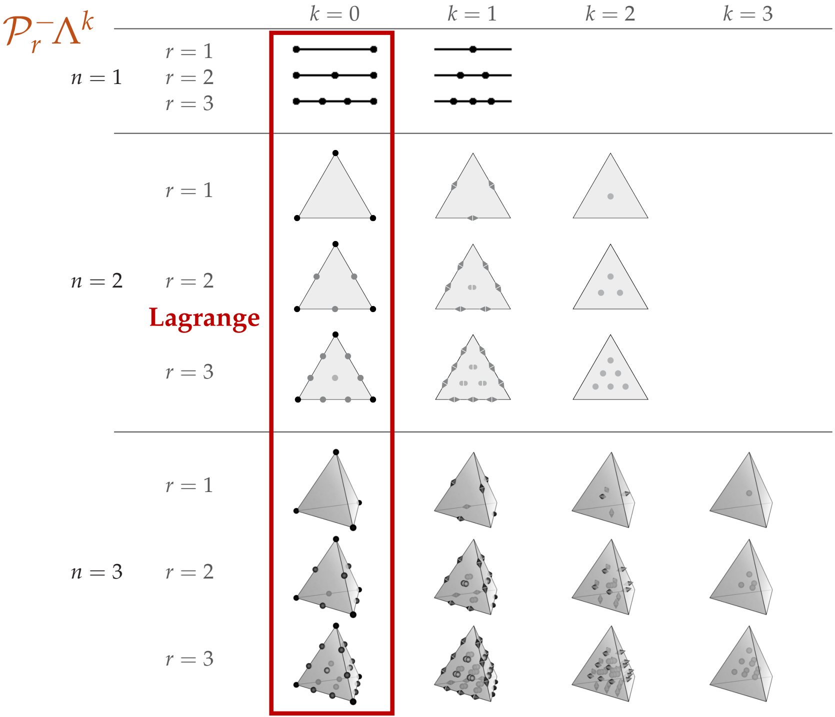
decreasing degree constant degree

- This leads to four stable discretizations of the mixed k-form Laplacian:

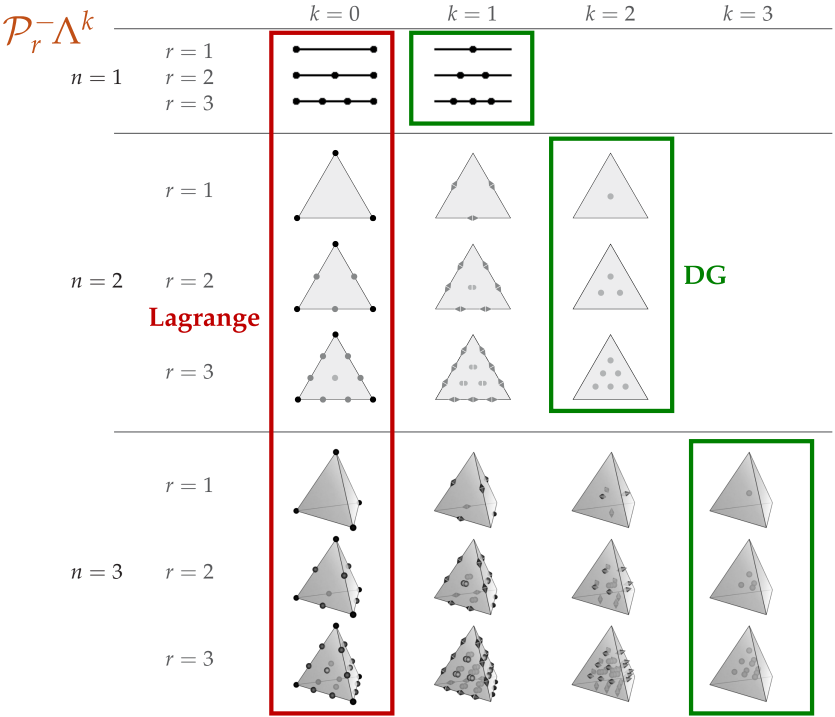
$$\mathcal{P}_r\Lambda^{k-1}(\mathcal{T}_h) \times \mathcal{P}_{r-1}\Lambda^k(\mathcal{T}_h) \qquad \mathcal{P}_r\Lambda^{k-1}(\mathcal{T}_h) \times \mathcal{P}_r^-\Lambda^k(\mathcal{T}_h)$$

$$\mathcal{P}_r^-\Lambda^{k-1}(\mathcal{T}_h) \times \mathcal{P}_{r-1}\Lambda^k(\mathcal{T}_h) \qquad \mathcal{P}_r^-\Lambda^{k-1}(\mathcal{T}_h) \times \mathcal{P}_r^-\Lambda^k(\mathcal{T}_h)$$


$\mathcal{P}_r^-\Lambda^k$
$k = 0$
$k = 1$
$k = 2$
$k = 3$
$n = 1$
$n = 2$
$n = 3$
$r = 1$
$r = 2$
$r = 3$



$\mathcal{P}_r^-\Lambda^k$
$k = 0$
$k = 1$
$k = 2$
$k = 3$
$n = 1$
$n = 2$
$n = 3$
$r = 1$
$r = 2$
$r = 3$
Lagrange



$\mathcal{P}_r^-\Lambda^k$
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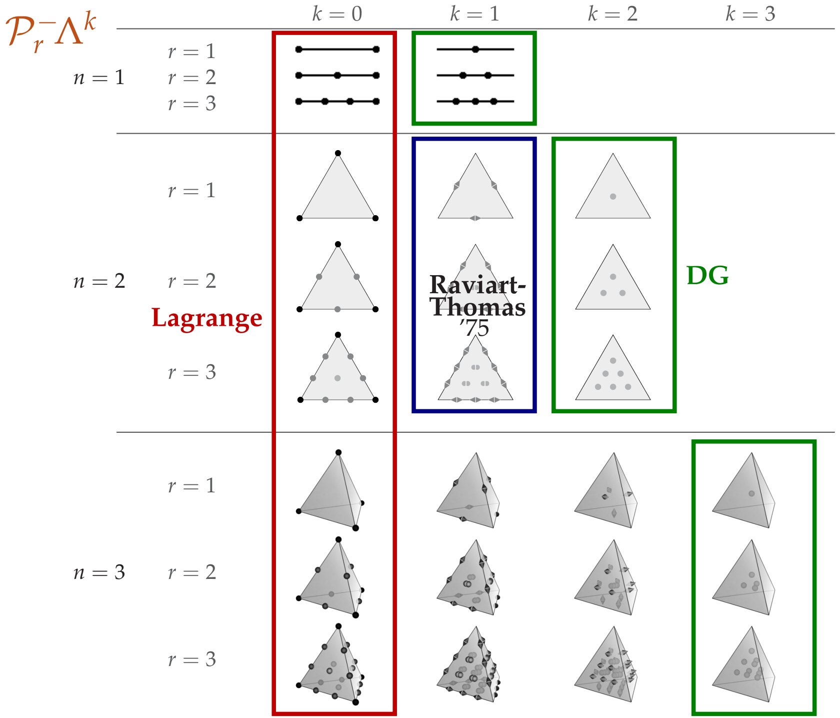
Lagrange
DG



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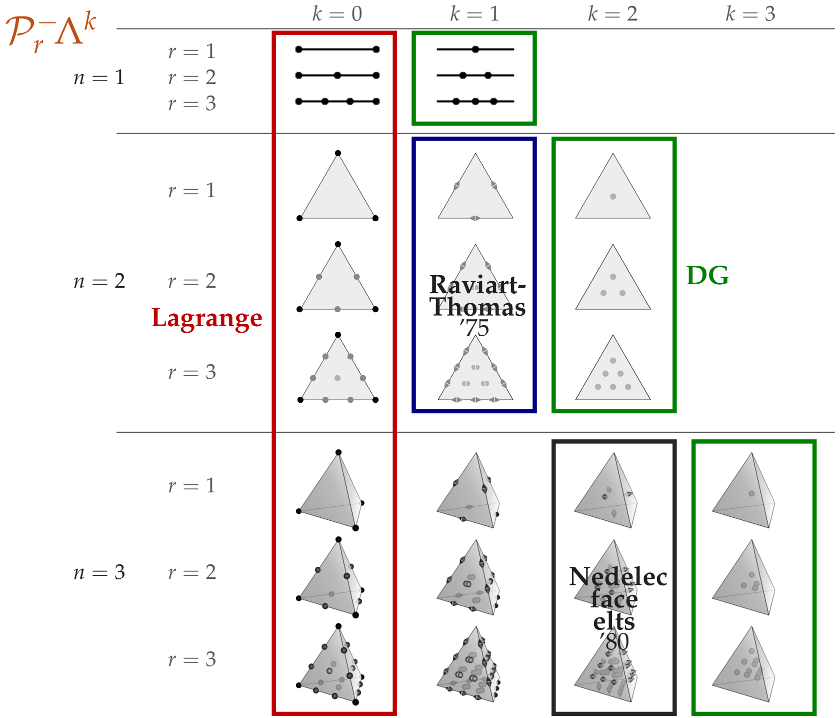
$r=1$
$r=2$
$r=3$
Lagrange
Raviart-Thomas '75
DG



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$n=1$
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$r=1$
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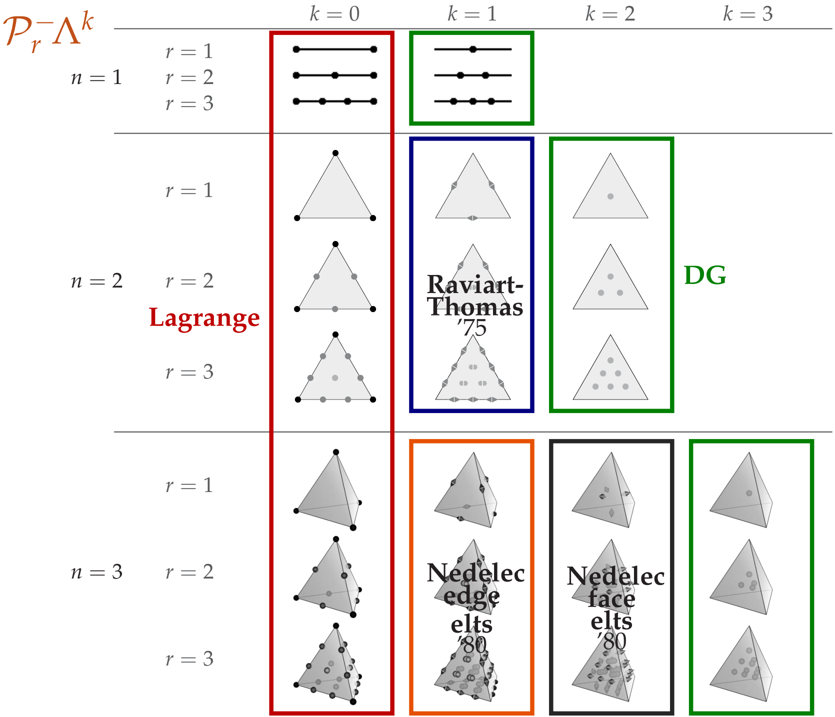
Lagrange
Raviart-Thomas '75
DG
Nedelec face elts '80



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Nedelec edge elts '80
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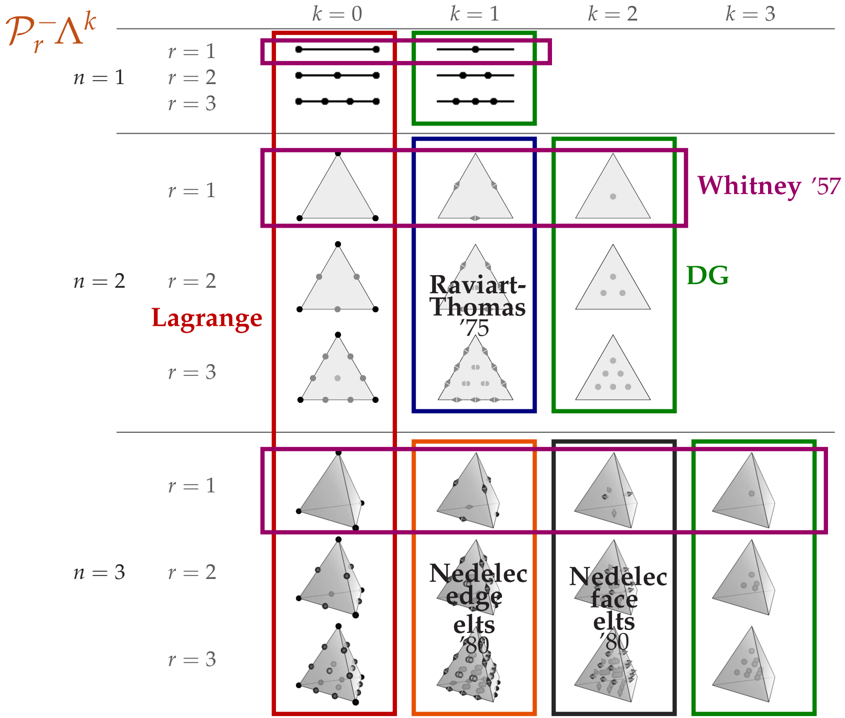
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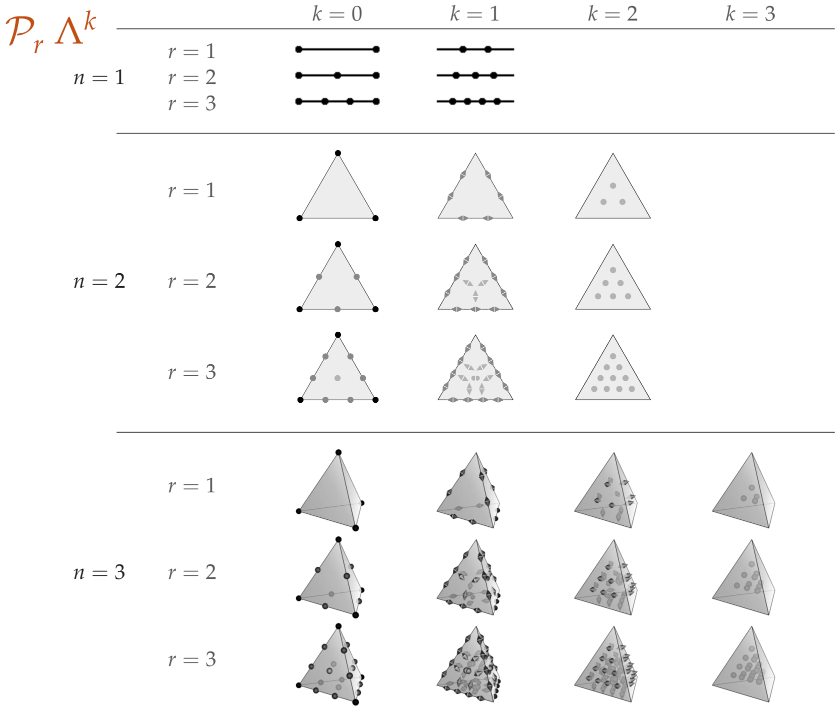
Whitney '57
DG
Lagrange
Raviart-Thomas '75
Nedelec edge elts '80
Nedelec face elts '80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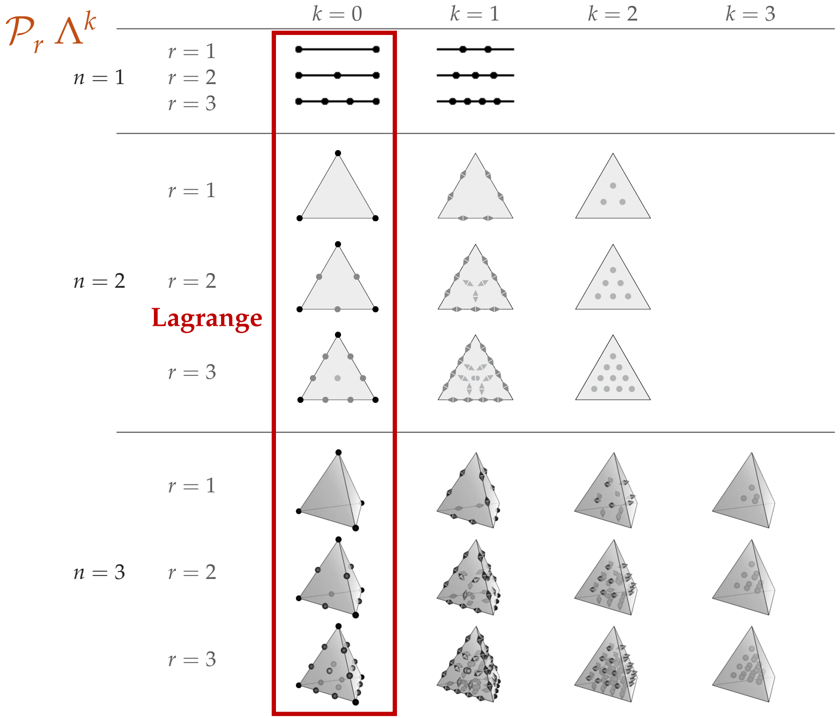
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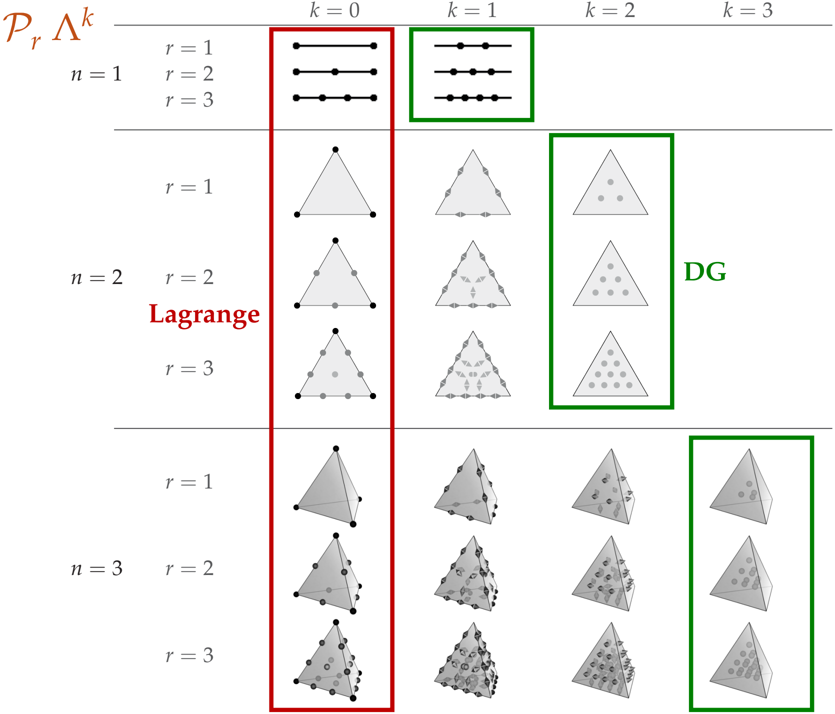
Lagrange



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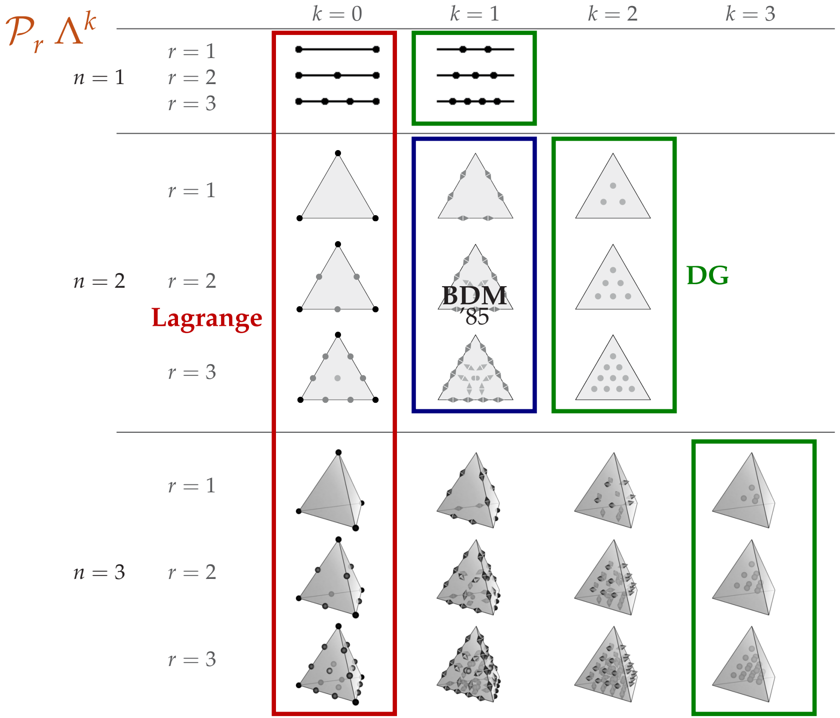
$r=3$
Lagrange
DG



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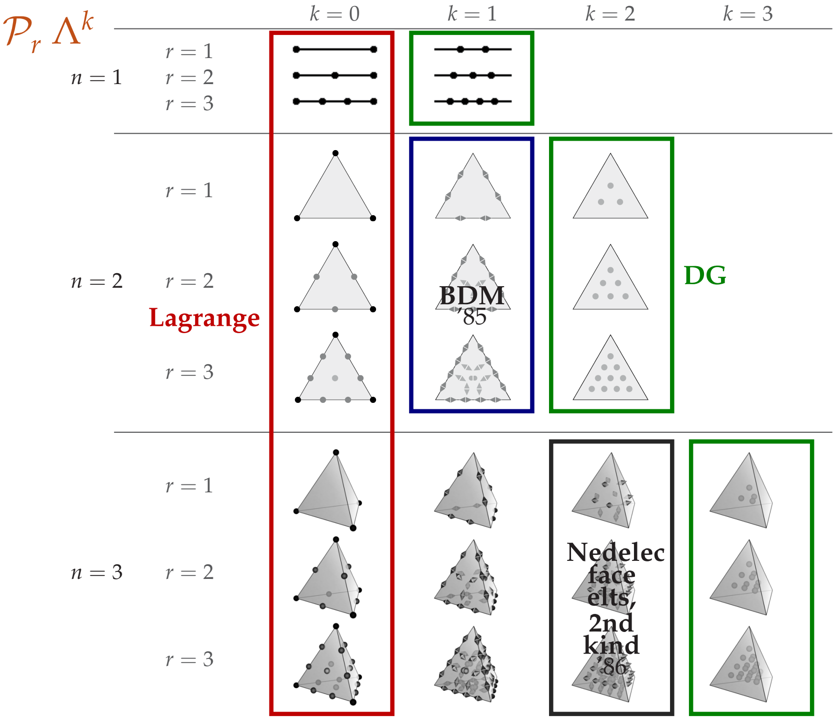
$r = 1$
$r = 2$
$r = 3$
Lagrange
BDM '85
DG



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$k = 3$
$n = 1$
$n = 2$
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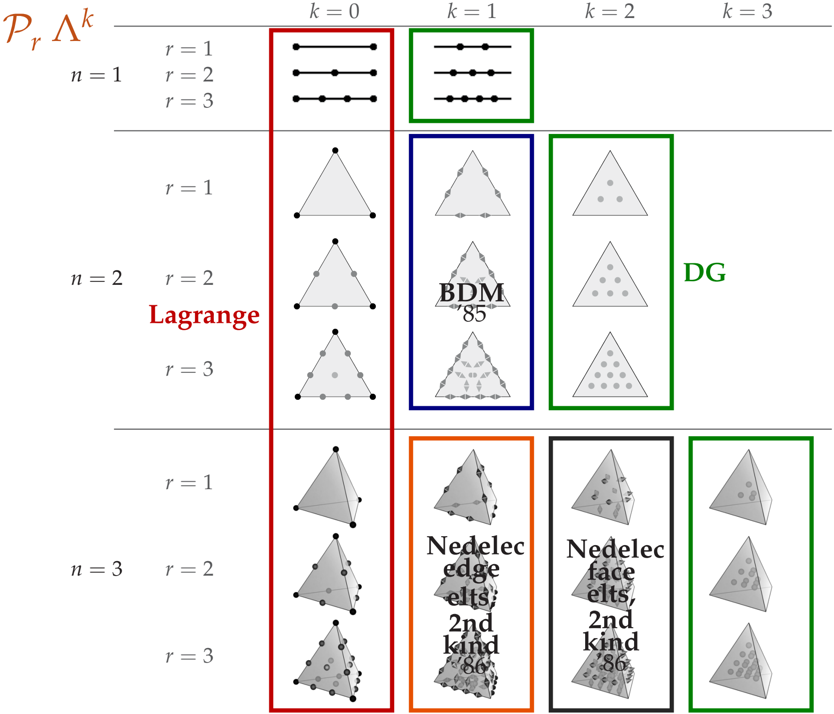
$r = 1$
$r = 2$
$r = 3$
Lagrange
BDM '85
DG
Nedelec face elts, 2nd kind '86



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$n=2$
$n=3$
$r=1$
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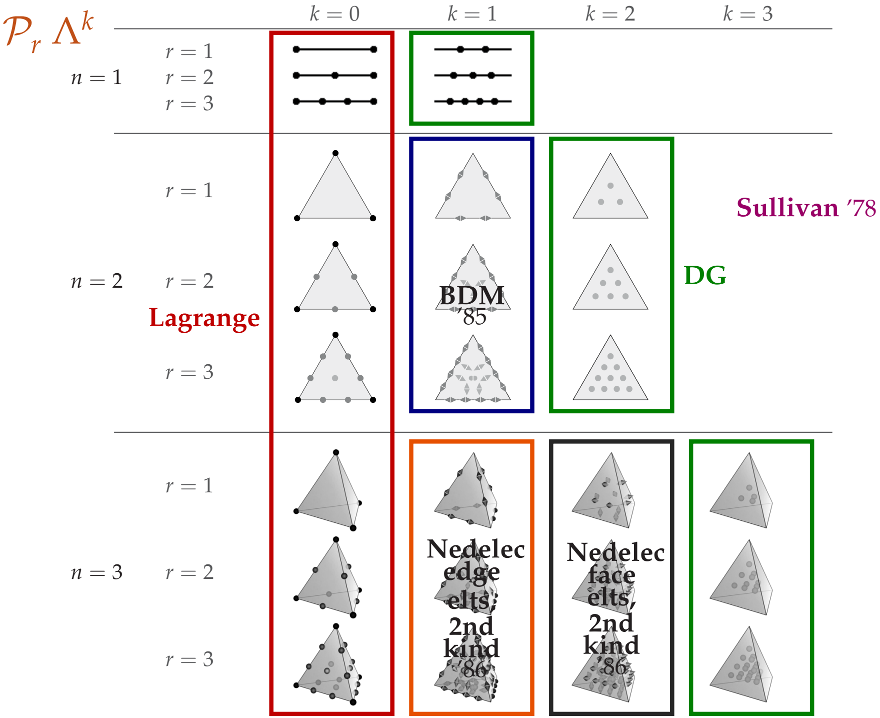
Lagrange
BDM '85
DG
Nedelec edge elts, 2nd kind '86
Nedelec face elts, 2nd kind '86



$\mathcal{P}_r \Lambda^k$
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$k = 2$
$k = 3$
$n = 1$
$r = 1$
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$r = 3$
$n = 2$
$r = 1$
$r = 2$
$r = 3$
$n = 3$
$r = 1$
$r = 2$
$r = 3$
Lagrange
BDM '85
DG
Sullivan '78
Nedelec edge elts, 2nd kind '86
Nedelec face elts, 2nd kind '86

Finite element differential forms on cubical meshes

DNA–Boffi–Bonizzoni 2012

Suppose we have a de Rham subcomplex V on an element $S \subset \mathbb{R}^m$:

$$\cdots \to V^k \xrightarrow{d} V^{k+1} \to \cdots \qquad V^k \subset H\Lambda^k(S)$$

and another, W, on another element $T \subset \mathbb{R}^n$:

$$\cdots \to W^k \xrightarrow{d} W^{k+1} \to \cdots$$

The tensor-product construction produces a new complex $V \wedge W$, a subcomplex of the de Rham complex on $S \times T$.

The tensor product construction

DNA–Boffi–Bonizzoni 2012

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$$\cdots \to W^k \xrightarrow{d} W^{k+1} \to \cdots$$

The tensor-product construction produces a new complex $V \wedge W$, a subcomplex of the de Rham complex on $S \times T$.

Shape fns: $\quad (V \wedge W)^k = \bigoplus_{i+j=k} \pi_S^* V^i \wedge \pi_T^* W^j \qquad (\pi_S : S \times T \to S)$

DOFs: $\quad (\eta \wedge \rho)(\pi_S^* v \wedge \pi_T^* w) := \eta(v)\rho(w)$

Finite element differential forms on cubes: the $\mathcal{Q}_r^-\Lambda^k$ family

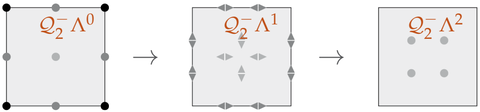
Start with the simple 1-D degree r finite element de Rham complex, V_r:

$$0 \;\to\; \mathcal{P}_r\Lambda^0(I) \;\xrightarrow{d}\; \mathcal{P}_{r-1}\Lambda^1(I) \;\to\; 0$$

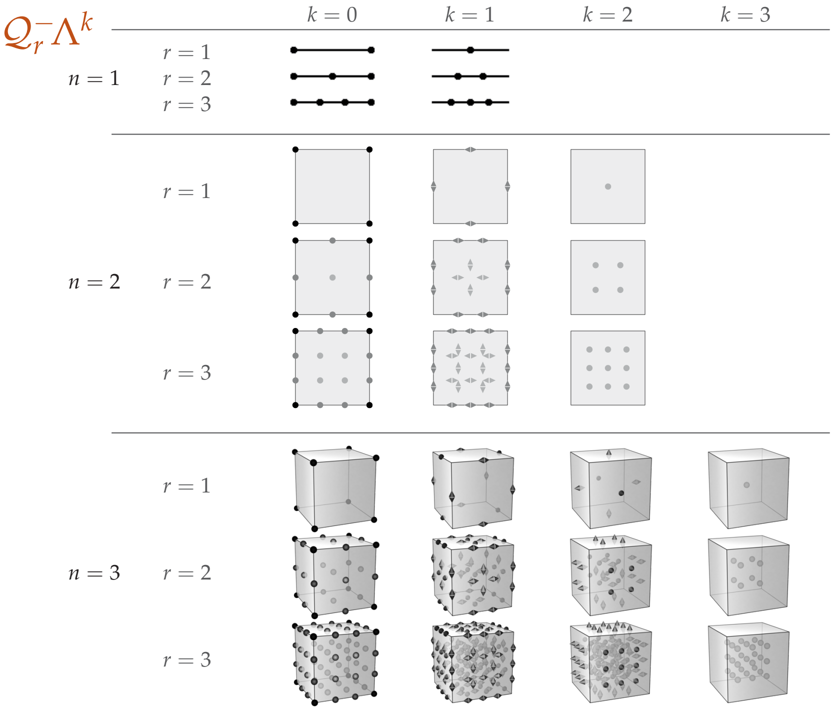
$$u(x) \quad\to\quad u'(x)\,dx$$

Take tensor product n times: $\quad \mathcal{Q}_r^-\Lambda^k(I^n) := (V_r \wedge \cdots \wedge V_r)^k$

$$\mathcal{Q}_r^-\Lambda^0 = \mathcal{Q}_r,$$
$$\mathcal{Q}_r^-\Lambda^1 = \mathcal{Q}_{r-1,r,r,\ldots}dx^1 + \mathcal{Q}_{r,r-1,r,\ldots}dx^2 + \cdots,$$
$$\mathcal{Q}_r^-\Lambda^2 = \mathcal{Q}_{r-1,r-1,r,\ldots}dx^1 \wedge dx^2 + \cdots, \quad \ldots$$



constant degree


$\mathcal{Q}_r^-\Lambda^k$
$k=0$
$k=1$
$k=2$
$k=3$
$n=1$
$n=2$
$n=3$
$r=1$
$r=2$
$r=3$

The 2nd family on cubes: 0-forms

DNA–Awanou 2011

The $\mathcal{Q}_r^- \Lambda^k$ family reduces to $\mathcal{Q}_r$ when $k = 0$. For the second family, we get the serendipy space $\mathcal{S}_r$.

The 2nd family on cubes: 0-forms

DNA–Awanou 2011

The $\mathcal{Q}_r^-\Lambda^k$ family reduces to $\mathcal{Q}_r$ when $k=0$. For the second family, we get the serendipy space $\mathcal{S}_r$.

2-D shape fns: $\mathcal{S}_r(I^2) = \mathcal{P}_r(I^2) \oplus \operatorname{span}[x_1^r x_2, x_1 x_2^r]$

DOFs: $u \mapsto \int_f \operatorname{tr}_f u\, q, \quad q \in \mathcal{P}_{r-2d}(f), f \in \Delta_d(I^n)$

The 2nd family on cubes: 0-forms

DNA–Awanou 2011

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DOFs: $u \mapsto \int_f \operatorname{tr}_f u\, q, \quad q \in \mathcal{P}_{r-2d}(f), f \in \Delta_d(I^n)$

n-D shape fns: $\mathcal{S}_r(I^n) = \mathcal{P}_r(I^n) \oplus \bigoplus_{\ell \geq 1} \mathcal{H}_{r+\ell,\ell}(I^n)$

$\mathcal{H}_{r,\ell}(I^n) =$ span of monomials of degree r, linear in $\geq \ell$ variables

DNA–Awanou 2012

The $\mathcal{S}_r\Lambda^k(I^n)$ family of FEDFs, uses the serendipity spaces for 0-forms, and serendipity-like DOFs.

DOFs: $u \mapsto \int_f \mathrm{tr}_f\, u \wedge q, \quad q \in \mathcal{P}_{r-2(d-k)}\Lambda^{d-k}(f), f \in \Delta_d(I^n), d \geq k$

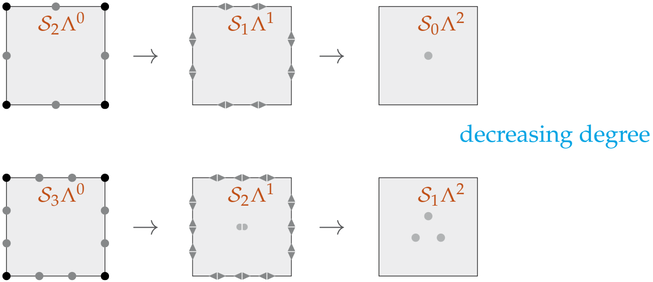
Shape fns:

$$\mathcal{S}_r\Lambda^k(I^n) = \mathcal{P}_r\Lambda^k(I^n) \oplus \bigoplus_{\ell \geq 1} \underbrace{[\kappa\mathcal{H}_{r+\ell-1,\ell}\Lambda^{k+1}(I^n) \oplus d\kappa\mathcal{H}_{r+\ell,\ell}\Lambda^k(I^n)]}_{\deg = r+\ell}$$

$$\mathcal{H}_{r,\ell}\Lambda^k(I^n) = \text{ span of monomials } x_1^{\alpha_i} \cdots x_n^{\alpha_n}\, dx_{\sigma_1} \wedge \cdots \wedge dx_{\sigma_k},$$
$$|\alpha| = r, \text{linear in} \geq \ell \text{ variables not counting the } x_{\sigma_i}$$

Unisolvence holds for all $n \geq 1$, $r \geq 1$, $0 \leq k \leq n$.

The 2nd cubic family in 2-D



$\mathcal{S}_r\Lambda^k(I^2)$

k	1	2	3	4	5
0	4	8	12	17	23
1	8	14	22	32	44
2	3	6	10	15	21

$\mathcal{Q}_r^-\Lambda^k(I^2)$

k	1	2	3	4	5
0	4	9	16	25	36
1	4	12	24	40	60
2	1	4	9	16	25

The 3D shape functions in traditional FE language

$\mathcal{S}_r\Lambda^0$: polynomials u such that $\deg u \le r + \operatorname{ldeg} u$

$\mathcal{S}_r\Lambda^1$:

$$(v_1, v_2, v_3) + (x_2 x_3(w_2 - w_3), x_3 x_1(w_3 - w_1), x_1 x_2(w_1 - w_2)) + \operatorname{grad} u,$$

$v_i \in \mathcal{P}_r, \quad w_i \in \mathcal{P}_{r-1}$ independent of $x_i, \quad \deg u \le r + \operatorname{ldeg} u + 1$

$\mathcal{S}_r\Lambda^2$:

$$(v_1, v_2, v_3) + \operatorname{curl}(x_2 x_3(w_2 - w_3), x_3 x_1(w_3 - w_1), x_1 x_2(w_1 - w_2)),$$

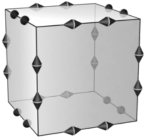
$v_i, w_i \in \mathcal{P}_r(I^3)$ with w_i independent of x_i

$\mathcal{S}_r\Lambda^3$: $v \in \mathcal{P}_r$

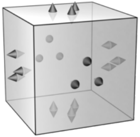
Dimensions and low order cases

k	$\mathcal{S}_r\Lambda^k(I^3)$ 1	2	3	4	5
0	8	20	32	50	74
1	24	48	84	135	204
2	18	39	72	120	186
3	4	10	20	35	56

k	$\mathcal{Q}_r^-\Lambda^k(I^3)$ 1	2	3	4	5
0	8	27	64	125	216
1	12	54	96	200	540
2	6	36	108	240	450
3	1	8	27	64	125



$\mathcal{S}_1\Lambda^1(I^3)$
new element

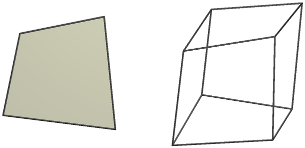


$\mathcal{S}_1\Lambda^2(I^3)$
corrected element

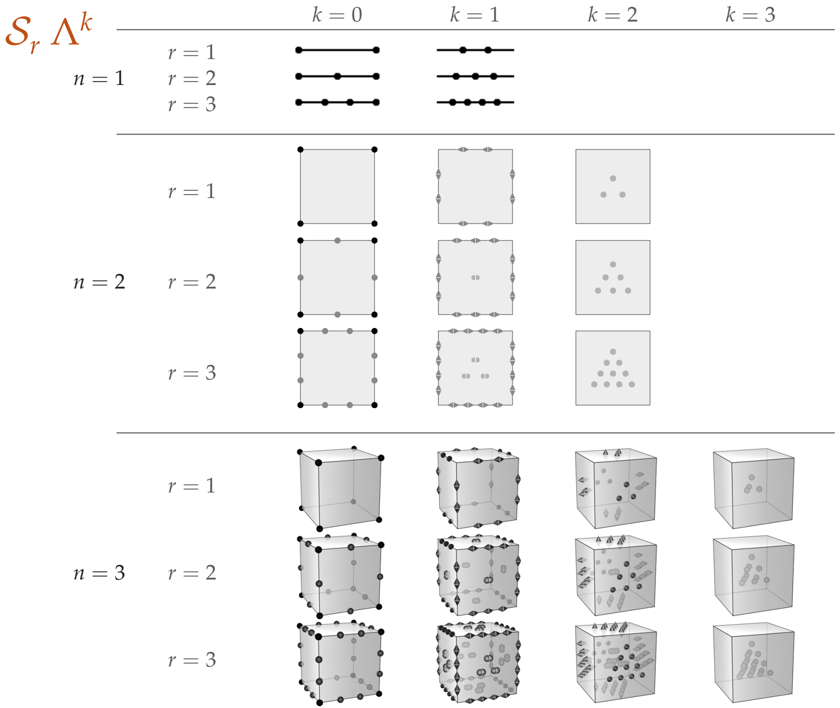
Approximation properties

On cubes the $\mathcal{Q}_r^-\Lambda^k$ and $\mathcal{S}_r^-\Lambda^k$ spaces provide the expected order of approximation. Same is true on parallelotopes, but accuracy is lost by non-affine distortions, *with greater loss, the greater the form degree k*.

- The L^2 approximation rate of the space $\mathcal{Q}_r = \mathcal{Q}_r^-\Lambda^0$ is $r+1$ on either affinely or multilinearly mapped elements.
- The rate for $\mathcal{S}_r = \mathcal{S}_r\Lambda^0$ is $r+1$ on affinely mapped elements, but only $\max(2, \lfloor r/n \rfloor + 1)$ on multilinearly mapped elements.
- The rate for $\mathcal{Q}_r^-\Lambda^k$, $k > 0$, is r on affinely mapped elements, $r-k+1$ on multilinearly mapped elements.
- The rate for $\mathcal{P}_r\Lambda^n = \mathcal{S}_r\Lambda^n$ is $r+1$ for affinely mapped elements, $\lfloor r/n \rfloor - n + 2$ for multilinearly mapped.



DNA-Boffi-Bonizzoni 2012



$\mathcal{S}_r \Lambda^k$
$k=0$
$k=1$
$k=2$
$k=3$
$n=1$
$n=2$
$n=3$
$r=1$
$r=2$
$r=3$

Periodic Table of the Finite Elements

$\mathcal{P}_r^-\Lambda^k$

The shape function space for $\mathcal{P}_r^-\Lambda^k$ is

$$\mathcal{P}_{r-1}\Lambda^k + \kappa\mathcal{P}_{r-1}\Lambda^{k+1},$$

where κ is the Koszul differential.[7] It includes the full polynomial space $\mathcal{P}_{r-1}\Lambda^k$, is included in $\mathcal{P}_r\Lambda^k$, and has dimension

$$\dim \mathcal{P}_r^-\Lambda^k(\Delta_n) = \binom{r+n}{r+k}\binom{r+k-1}{k}.$$

The degrees of freedom are given on faces f of dimension $d \geq k$ by moments of the trace weighted by a full polynomial space:

$$u \mapsto \int_f (\mathrm{tr}_f\, u) \wedge q, \quad q \in \mathcal{P}_{r+k-d-1}\Lambda^{d-k}(f).$$

The spaces with constant degree r form a complex:

$$\mathcal{P}_r^-\Lambda^0 \xrightarrow{d} \mathcal{P}_r^-\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_r^-\Lambda^n.$$

$\mathcal{P}_r\Lambda^k$

The shape function space for $\mathcal{P}_r\Lambda^k$ consists of all differential k-forms with polynomial coefficients of degree at most r, and has dimension

$$\dim \mathcal{P}_r\Lambda^k(\Delta_n) = \binom{r+n}{r+k}\binom{r+k}{k}.$$

The degrees of freedom are given on faces f of dimension $d \geq k$ by moments of the trace weighted by a $\mathcal{P}_r^-$ space:

$$u \mapsto \int_f (\mathrm{tr}_f\, u) \wedge q, \quad q \in \mathcal{P}_{r+k-d}^-\Lambda^{d-k}(f).$$

The spaces with decreasing degree r form a complex:

$$\mathcal{P}_r\Lambda^0 \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_{r-n}\Lambda^n.$$

$\mathcal{Q}_r^-\Lambda^k$

This family is constructed from the complex of 1-dimensional finite elements using a tensor product construction.[10] The shape function space on the unit cube $\square_n = I^n$ is given by

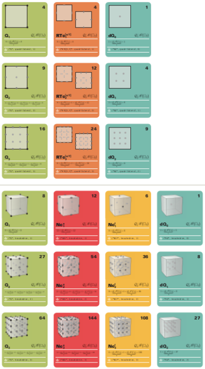
$$\bigoplus_{\sigma\in\Sigma(k,n)} \left[\bigotimes_{i=1}^{n} \mathcal{P}_{r-\lambda_{i,\sigma}}(I)\right] dx^{\sigma_1}\wedge\cdots\wedge dx^{\sigma_k},$$

where $\Sigma(k,n)$ denotes the increasing maps $\{1,\dots,k\} \to \{1,\dots,n\}$. Its dimension is $\dim \mathcal{Q}_r^-\Lambda^k(\square_n) = \binom{n}{k} r^k (r+1)^{n-k}$. The degrees of freedom are

$$u \mapsto \int_f (\mathrm{tr}_f\, u) \wedge q, \quad q \in \mathcal{Q}_{r-1}^-\Lambda^{d-k}(f).$$

The spaces with constant degree r form a complex:

$$\mathcal{Q}_r^-\Lambda^0 \xrightarrow{d} \mathcal{Q}_r^-\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{Q}_r^-\Lambda^n.$$



$\mathcal{S}_r\Lambda^k$

The shape function space for $\mathcal{S}_r\Lambda^k$ is given by

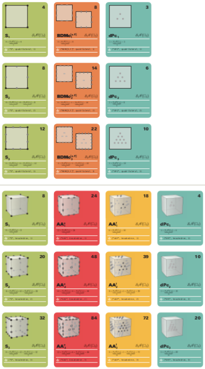
$$\mathcal{P}_r\Lambda^k \oplus \bigoplus_{\ell\geq 1}[\kappa\mathcal{H}_{r+\ell-1,\ell}\Lambda^{k+1} \oplus d\kappa\mathcal{H}_{r+\ell,\ell}\Lambda^k],$$

where $\mathcal{H}_{r,\ell}\Lambda^k$ consists of homogeneous polynomial k-forms of degree r which are linear and undifferentiated in at least ℓ variables.[11] Its dimension is $\dim \mathcal{S}_r\Lambda^k(\square_n) = \sum_{d\geq k} 2^{n-d}\binom{n}{d}\binom{r-d+2k}{d}\binom{d}{k}$. The degrees of freedom are

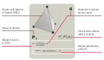
$$u \mapsto \int_f (\mathrm{tr}_f\, u) \wedge q, \quad q \in \mathcal{P}_{r-2(d-k)}\Lambda^{d-k}(f).$$

The spaces with decreasing degree r form a complex:

$$\mathcal{S}_r\Lambda^0 \xrightarrow{d} \mathcal{S}_{r-1}\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{S}_{r-n}\Lambda^n.$$



Legend



Finite elements

The table presents the primary spaces of finite elements for the discretization of the fundamental operators of vector calculus: the gradient, curl, and divergence. A finite element space is a space of piecewise polynomial functions on a domain determined by: (1) a mesh of the domain into polyhedral cells called *elements*; (2) a finite dimensional space of polynomial functions on each element called the *shape functions*; and (3) a unisolvent set of functionals on the shape functions of each element called *degrees of freedom* (DOFs), each DOF being associated to a (generalized) face of the element, and specifying a quantity which takes a single value for all elements sharing the face. The element diagrams depict the DOFs and their association to faces.

The spaces $\mathcal{P}_r^-\Lambda^k$ and $\mathcal{P}_r\Lambda^k$ depicted on the left half of the table are the two primary families of finite element spaces for meshes of simplices, and the spaces $\mathcal{Q}_r^-\Lambda^k$ and $\mathcal{S}_r\Lambda^k$ on the right side are for meshes of cubes or boxes. Each is defined in any dimension $n \geq 1$ for each value of the polynomial degree $r \geq 1$, and each value of $0 \leq k \leq n$. The parameter k refers to the operator: the spaces consist of differential k-forms which belong to the domain of the kth exterior derivative.

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Periodic Table of the Finite Elements

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New complexes from old

Elasticity with weak symmetry

The mixed formulation of elasticity with *weak symmetry* is more amenable to discretization than the standard mixed formulation.

Fraeijs de Veubeke '75

$$p = \operatorname{skw}\operatorname{grad} u, \quad A\sigma = \operatorname{grad} u - p$$

Find $\quad \sigma \in L^2(\Omega) \otimes \mathbb{R}^{n\times n}, u \in L^2(\Omega) \otimes \mathbb{R}^n, p \in L^2(\Omega) \otimes \mathbb{R}^{n\times n}_{\text{skw}} \quad$ s.t.

$$\begin{aligned} \langle A\sigma, \tau\rangle + \langle u, \operatorname{div}\tau\rangle + \langle p, \tau\rangle &= 0, && \tau \in L^2(\Omega) \otimes \mathbb{R}^{n\times n} \\ -\langle \operatorname{div}\sigma, v\rangle &= \langle f, v\rangle, && v \in L^2(\Omega) \otimes \mathbb{R}^n \\ -\langle \sigma, q\rangle &= 0, && q \in L^2(\Omega) \otimes \mathbb{R}^{n\times n}_{\text{skw}} \end{aligned}$$

This is exactly the mixed Hodge Laplacian for the complex:

$$L^2_A(\Omega) \otimes \mathbb{R}^{n\times n} \xrightarrow{(-\operatorname{div}, -\operatorname{skw})} [L^2(\Omega) \otimes \mathbb{R}^n] \oplus [L^2(\Omega) \otimes \mathbb{R}^{n\times n}_{\text{skw}}] \longrightarrow 0$$

Well-posedness

$$L^2_A(\Omega) \otimes \mathbb{R}^{n\times n} \xrightarrow{(-\operatorname{div}, -\operatorname{skw})} [L^2(\Omega)\otimes\mathbb{R}^n] \oplus [L^2(\Omega)\otimes\mathbb{R}^{n\times n}_{\operatorname{skw}}] \longrightarrow 0$$

Well-posedness depends on the exactness of the complex. This can be shown by relating the complex to two de Rham complexes:

$$\begin{tikzcd}
 & L^2(\Omega)\otimes\mathbb{R}^n\otimes\mathbb{R}^{n\times n}_{\operatorname{skw}} \arrow[r, "\operatorname{div}"] & L^2(\Omega)\otimes\mathbb{R}^{n\times n}_{\operatorname{skw}} \arrow[r] & 0 \\
L^2(\Omega)\otimes\mathbb{R}^{n\times n} \arrow[ur, "S"] \arrow[r, "\operatorname{curl}"] & L^2(\Omega)\otimes\mathbb{R}^{n\times n} \arrow[ur, "-\operatorname{skw}"] \arrow[r, "-\operatorname{div}"] & L^2(\Omega)\otimes\mathbb{R}^n \arrow[r] & 0
\end{tikzcd}$$

$S\tau = \tau^T - \operatorname{tr}(\tau)I$ (invertible)

Well-posedness

$$L^2_A(\Omega) \otimes \mathbb{R}^{n\times n} \xrightarrow{(-\operatorname{div}, -\operatorname{skw})} [L^2(\Omega) \otimes \mathbb{R}^n] \oplus [L^2(\Omega) \otimes \mathbb{R}^{n\times n}_{\operatorname{skw}}] \longrightarrow 0$$

Well-posedness depends on the exactness of the complex. This can be shown by relating the complex to two de Rham complexes:

$$\begin{tikzcd}
 & & q & \\
 & L^2(\Omega) \otimes \mathbb{R}^n \otimes \mathbb{R}^{n\times n}_{\operatorname{skw}} \arrow[r, "\operatorname{div}"] & L^2(\Omega) \otimes \mathbb{R}^{n\times n}_{\operatorname{skw}} \arrow[r] & 0 \\
L^2(\Omega) \otimes \mathbb{R}^{n\times n} \arrow[ur, "S"] \arrow[r, "\operatorname{curl}"] & L^2(\Omega) \otimes \mathbb{R}^{n\times n} \arrow[ur, "-\operatorname{skw}"] \arrow[r, "-\operatorname{div}"] & L^2(\Omega) \otimes \mathbb{R}^n \arrow[r] & 0 \\
 & & v &
\end{tikzcd}$$

$S\tau = \tau^T - \operatorname{tr}(\tau) I$ (invertible)

$$L^2_A(\Omega) \otimes \mathbb{R}^{n\times n} \xrightarrow{(-\operatorname{div}, -\operatorname{skw})} [L^2(\Omega) \otimes \mathbb{R}^n] \oplus [L^2(\Omega) \otimes \mathbb{R}^{n\times n}_{\mathrm{skw}}] \longrightarrow 0$$

Well-posedness depends on the exactness of the complex. This can be shown by relating the complex to two de Rham complexes:

$$\begin{tikzcd}
 & & q & \\
 & L^2(\Omega)\otimes\mathbb{R}^n\otimes\mathbb{R}^{n\times n}_{\mathrm{skw}} \arrow[r, "\operatorname{div}"] & L^2(\Omega)\otimes\mathbb{R}^{n\times n}_{\mathrm{skw}} \arrow[r] & 0 \\
L^2(\Omega)\otimes\mathbb{R}^{n\times n} \arrow[ur, "S"] \arrow[r, "\operatorname{curl}"] & L^2(\Omega)\otimes\mathbb{R}^{n\times n} \arrow[ur, "-\operatorname{skw}"] \arrow[r, "-\operatorname{div}"] & L^2(\Omega)\otimes\mathbb{R}^{n} \arrow[r] & 0 \\
 & \rho & v \arrow[l] &
\end{tikzcd}$$

$S\tau = \tau^T - \operatorname{tr}(\tau) I$ (invertible)

Well-posedness

$$L^2_A(\Omega)\otimes\mathbb{R}^{n\times n}\xrightarrow{(-\operatorname{div},-\operatorname{skw})}[L^2(\Omega)\otimes\mathbb{R}^n]\oplus[L^2(\Omega)\otimes\mathbb{R}^{n\times n}_{\operatorname{skw}}]\longrightarrow 0$$

Well-posedness depends on the exactness of the complex. This can be shown by relating the complex to two de Rham complexes:

$$\begin{tikzcd}
 & & q+\operatorname{skw}\rho & \\
 & L^2(\Omega)\otimes\mathbb{R}^n\otimes\mathbb{R}^{n\times n}_{\operatorname{skw}} \arrow[r, "\operatorname{div}"] & L^2(\Omega)\otimes\mathbb{R}^{n\times n}_{\operatorname{skw}} \arrow[r] & 0 \\
L^2(\Omega)\otimes\mathbb{R}^{n\times n} \arrow[ur, "S"] \arrow[r, "\operatorname{curl}"] & L^2(\Omega)\otimes\mathbb{R}^{n\times n} \arrow[ur, "-\operatorname{skw}"] \arrow[r, "-\operatorname{div}"] & L^2(\Omega)\otimes\mathbb{R}^n \arrow[r] & 0 \\
 & \rho \arrow[uuur] & v \arrow[l] &
\end{tikzcd}$$

$S\tau=\tau^T-\operatorname{tr}(\tau)I$ (invertible)

Well-posedness

$$L^2_A(\Omega)\otimes\mathbb{R}^{n\times n} \xrightarrow{(-\operatorname{div},-\operatorname{skw})} [L^2(\Omega)\otimes\mathbb{R}^n]\oplus[L^2(\Omega)\otimes\mathbb{R}^{n\times n}_{\operatorname{skw}}] \longrightarrow 0$$

Well-posedness depends on the exactness of the complex. This can be shown by relating the complex to two de Rham complexes:

$$\begin{tikzcd}
 & \psi & q+\operatorname{skw}\rho \arrow[l] & \\
 & L^2(\Omega)\otimes\mathbb{R}^n\otimes\mathbb{R}^{n\times n}_{\operatorname{skw}} \arrow[r, "\operatorname{div}"] & L^2(\Omega)\otimes\mathbb{R}^{n\times n}_{\operatorname{skw}} \arrow[r] & 0 \\
L^2(\Omega)\otimes\mathbb{R}^{n\times n} \arrow[ur, "S"] \arrow[r, "\operatorname{curl}"] & L^2(\Omega)\otimes\mathbb{R}^{n\times n} \arrow[ur, "-\operatorname{skw}"] \arrow[r, "-\operatorname{div}"] & L^2(\Omega)\otimes\mathbb{R}^n \arrow[r] & 0 \\
 & \rho \arrow[uuur] & v \arrow[l] &
\end{tikzcd}$$

$S\tau = \tau^T - \operatorname{tr}(\tau)I$ (invertible)

Well-posedness

$$L^2_A(\Omega)\otimes\mathbb{R}^{n\times n} \xrightarrow{(-\operatorname{div},-\operatorname{skw})} [L^2(\Omega)\otimes\mathbb{R}^n]\oplus[L^2(\Omega)\otimes\mathbb{R}^{n\times n}_{\mathrm{skw}}] \longrightarrow 0$$

Well-posedness depends on the exactness of the complex. This can be shown by relating the complex to two de Rham complexes:

$$\begin{tikzcd}
 & \psi \arrow[dddl] & & q+\operatorname{skw}\rho \arrow[ll] & & \\
 & L^2(\Omega)\otimes\mathbb{R}^n\otimes\mathbb{R}^{n\times n}_{\mathrm{skw}} \arrow[rr, "\operatorname{div}"] & & L^2(\Omega)\otimes\mathbb{R}^{n\times n}_{\mathrm{skw}} \arrow[rr] & & 0 \\
L^2(\Omega)\otimes\mathbb{R}^{n\times n} \arrow[ur, "S"] \arrow[rr, "\operatorname{curl}"] & & L^2(\Omega)\otimes\mathbb{R}^{n\times n} \arrow[ur, "-\operatorname{skw}"] \arrow[rr, "-\operatorname{div}"] & & L^2(\Omega)\otimes\mathbb{R}^n \arrow[r] & 0 \\
\phi & & \rho \arrow[uuur] & & v \arrow[ll] &
\end{tikzcd}$$

$S\tau = \tau^T - \operatorname{tr}(\tau)I \quad$ (invertible)

Well-posedness

$$L^2_A(\Omega) \otimes \mathbb{R}^{n\times n} \xrightarrow{(-\operatorname{div}, -\operatorname{skw})} [L^2(\Omega) \otimes \mathbb{R}^n] \oplus [L^2(\Omega) \otimes \mathbb{R}^{n\times n}_{\operatorname{skw}}] \longrightarrow 0$$

Well-posedness depends on the exactness of the complex. This can be shown by relating the complex to two de Rham complexes:

$$\begin{tikzcd}
 & \psi \arrow[dddl] & q + \operatorname{skw}\rho \arrow[l] & \\
 & L^2(\Omega) \otimes \mathbb{R}^n \otimes \mathbb{R}^{n\times n}_{\operatorname{skw}} \arrow[r, "\operatorname{div}"] & L^2(\Omega) \otimes \mathbb{R}^{n\times n}_{\operatorname{skw}} \arrow[r] & 0 \\
L^2(\Omega) \otimes \mathbb{R}^{n\times n} \arrow[ur, "S"] \arrow[r, "\operatorname{curl}"] & L^2(\Omega) \otimes \mathbb{R}^{n\times n} \arrow[ur, "-\operatorname{skw}"] \arrow[r, "-\operatorname{div}"] & L^2(\Omega) \otimes \mathbb{R}^n \arrow[r] & 0 \\
\phi \arrow[r] & \operatorname{curl}\phi + \rho \arrow[uuur] & v \arrow[l] &
\end{tikzcd}$$

$S\tau = \tau^T - \operatorname{tr}(\tau)I$ (invertible)

To discretize we select discrete de Rham subcomplexes with commuting projs

$$\tilde V_h^0 \xrightarrow{\text{curl}} \tilde V_h^1 \xrightarrow{-\text{div}} \tilde V_h^2 \to 0, \qquad V_h^1 \xrightarrow{-\text{div}} V_h^2 \to 0$$

to get the discrete complex

$$\tilde V_h^1 \otimes \mathbb{R}^n \xrightarrow{(-\operatorname{div}, -\operatorname{skw})} (\tilde V_h^2 \otimes \mathbb{R}^n) \times (V_h^2 \otimes \mathbb{R}^{n\times n}_{\text{skw}}) \to 0$$

We get stability if we can carry out the diagram chase on:

$$\begin{tikzcd}
 & V_h^1 \otimes \mathbb{R}^{n\times n}_{\text{skw}} \arrow[r, "\text{div}"] & V_h^2 \otimes \mathbb{R}^{n\times n}_{\text{skw}} \arrow[r] & 0 \\
\tilde V_h^0 \otimes \mathbb{R}^n \arrow[ur, "\pi_h^1 S"] \arrow[r, "\text{curl}"] & \tilde V_h^1 \otimes \mathbb{R}^n \arrow[ur, "-\pi_h^2 \operatorname{skw}"] \arrow[r, "-\text{div}"] & \tilde V_h^2 \otimes \mathbb{R}^n \arrow[r] & 0
\end{tikzcd}$$

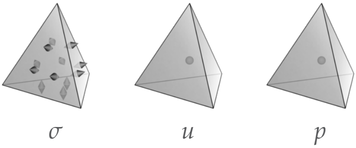
This requires that $\pi_h^1 S : \tilde V_h^0 \otimes \mathbb{R}^n \to V_h^1 \otimes \mathbb{R}^{n\times n}_{\text{skw}}$ is *surjective*.

Stable elements

The requirement that $\pi_h^1 S : \tilde{V}_h^0 \otimes \mathbb{R}^n \to V_h^1 \otimes \mathbb{R}^{n\times n}_{\text{skw}}$ is surjective can be checked by looking at DOFs.
The simplest choice is

$$\mathcal{P}_{r+1}\Lambda^{n-2} \xrightarrow{\text{curl}} \mathcal{P}_r\Lambda^{n-1} \xrightarrow{-\text{div}} \mathcal{P}_{r-1}\Lambda^n \to 0, \quad \mathcal{P}_r^-\Lambda^{n-1} \xrightarrow{\text{div}} \mathcal{P}_r^-\Lambda^n \to 0$$

which gives the elements of DNA–Falk–Winther ’07



Other elements:
Cockburn–Gopalakrishnan–Guzmán,
Gopalakrishnan–Guzmán, Stenberg, …

More complexes from complexes

$$\begin{tikzcd} 0 \arrow[r] & V^1 \arrow[r, "d"] & W^2 \\ 0 \arrow[r] & \tilde V^1 \arrow[r, "\tilde d"] \arrow[ur, "S"] & \tilde W^2 \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} 0 \arrow[r] & \Gamma \arrow[r, "D"] & \tilde W^2 \end{tikzcd}$$

where $\Gamma = \{(v,\tau) \in V^1 \times \tilde V^1 \,|\, dv = S\tau\}, \quad D(v,\tau) = \tilde d \tau.$

$$\begin{tikzcd} 0 \arrow[r] & V_h^1 \arrow[r, "d"] & V_h^2 \\ 0 \arrow[r] & \tilde V_h^1 \arrow[r, "\tilde d"] \arrow[ur, "\pi_h^2 S"] & \tilde W^2 \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} 0 \arrow[r] & \Gamma_h \arrow[r, "D"] & \tilde W^2 \end{tikzcd}$$

where $\Gamma_h = \{(v,\tau) \in V_h^1 \times \tilde V_h^1 \,|\, dv = \pi_h^2 S\tau\}.$

Find $\quad u_h \in V_h^1, \sigma_h \in \tilde V_h^1, \lambda_h \in V_h^2 \quad$ s.t.

$$\langle \tilde d\sigma_h, \tilde d\tau\rangle + \langle \lambda_h, dv - \pi_h S\tau\rangle = \langle f, v\rangle, \quad v \in V_h^1,\ \tau \in \tilde V_h^1,$$
$$\langle du_h - \pi_h S\sigma_h, \mu\rangle = 0, \qquad \mu \in V_h^2.$$

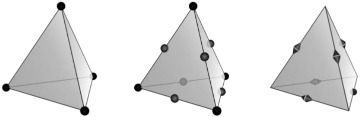
FEEC discretization of the biharmonic

$$\begin{tikzcd} 0 \arrow[r] & \mathring{H}^1(\Omega) \arrow[r, "\mathrm{grad}"] & L^2(\Omega;\mathbb{R}^n) \\ 0 \arrow[r] & \mathring{H}^1(\Omega;\mathbb{R}^n) \arrow[r, "\mathrm{grad}"] \arrow[ur, "I"] & L^2_C(\Omega;\mathbb{R}^{n\times n}) \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{P}_r\Lambda^0 \arrow[r, "\mathrm{grad}"] & \mathcal{P}_r^-\Lambda^1 \\ 0 \arrow[r] & \mathcal{P}_{r+1}\Lambda^0 \otimes \mathbb{R}^n \arrow[r, "\mathrm{grad}"] \arrow[ur, "\pi_h"] & L^2_C(\Omega;\mathbb{R}^{n\times n}) \end{tikzcd}$$

Find $\quad u_h \in \mathcal{P}_r\Lambda^0, \sigma_h \in \mathcal{P}_{r+1}\Lambda^0, \lambda_h \in \mathcal{P}_r^-\Lambda^1 \quad$ s.t.

$$\begin{aligned} \langle C\,\mathrm{grad}\,\sigma_h, \mathrm{grad}\,\tau\rangle + \langle \lambda_h, \mathrm{grad}\,v - \pi_h\tau\rangle &= \langle f, v\rangle, & v \in \mathcal{P}_r\Lambda^0,\ \tau \in \mathcal{P}_{r+1}\Lambda^0, \\ \langle \mathrm{grad}\,u_h - \pi_h\sigma_h, \mu\rangle &= 0, & \mu \in \mathcal{P}_r^-\Lambda^1. \end{aligned}$$



u $\quad\quad$ σ $\quad\quad$ λ

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DNA, Falk, Winther, Finite element exterior calculus: from Hodge theory to numerical stability. AMS Bulletin 2010, p. 281-354

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