A Review of Solution Techniques for Unsteady Incompressible Flow

David Silvester
School of Mathematics
University of Manchester

Outline

- PDEs
- Review : 1966 – 1999
- Update : 2000 – 2009

Outline

- PDEs
  -

$$\left.\begin{aligned} \frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} - \nu \nabla^2 \vec{u} + \nabla p = 0 \\ \nabla \cdot \vec{u} = 0 \end{aligned}\right\} \quad \text{Navier–Stokes}$$

Outline

- PDEs
  - $$\left.\begin{aligned} \frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} - \nu \nabla^2 \vec{u} + \nabla p = 0 \\ \nabla \cdot \vec{u} = 0 \end{aligned}\right\}$$ Navier–Stokes
  - $$\left.\begin{aligned} \frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} - \nu \nabla^2 \vec{u} + \nabla p = \vec{j} T \\ \nabla \cdot \vec{u} = 0 \\ \frac{\partial T}{\partial t} + \vec{u} \cdot \nabla T - \nu \nabla^2 T = 0 \end{aligned}\right\}$$ Boussinesq

Navier-Stokes Equations

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} - \nu \nabla^2 \vec{u} + \nabla p = 0 \qquad \text{in } \mathcal{W} \equiv \Omega \times (0, T)$$

$$\nabla \cdot \vec{u} = 0 \qquad \text{in } \mathcal{W}$$

Boundary and Initial conditions

$$\begin{aligned} \vec{u} &= \vec{g} \quad && \text{on } \Gamma_D \times [0, T]; \\ \nu \nabla \vec{u} \cdot \vec{n} - p \vec{n} &= \vec{0} \quad && \text{on } \Gamma_N \times [0, T]; \\ \vec{u}(\vec{x}, 0) &= \vec{u}_0(\vec{x}) \quad && \text{in } \Omega. \end{aligned}$$

Finite element matrix formulation

Introducing the basis sets

$$\mathbf{X}_h = \text{span}\{\vec{\phi}_i\}_{i=1}^{n_u}, \quad \text{Velocity basis functions;}$$
$$M_h = \text{span}\{\psi_j\}_{j=1}^{n_p}, \quad \text{Pressure basis functions.}$$

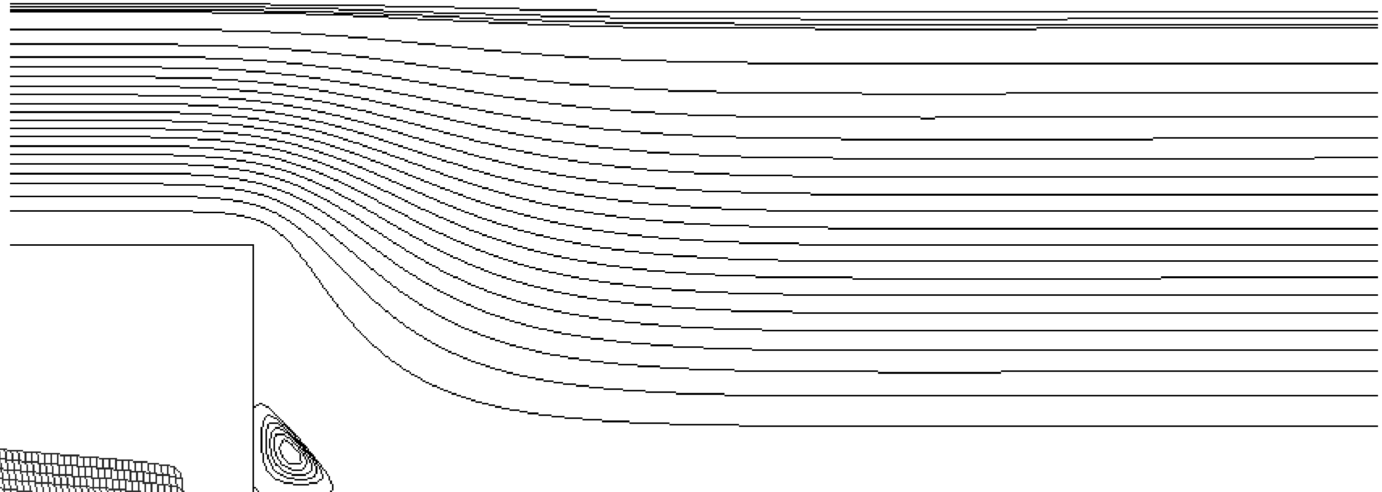
gives the method-of-lines discretized system:

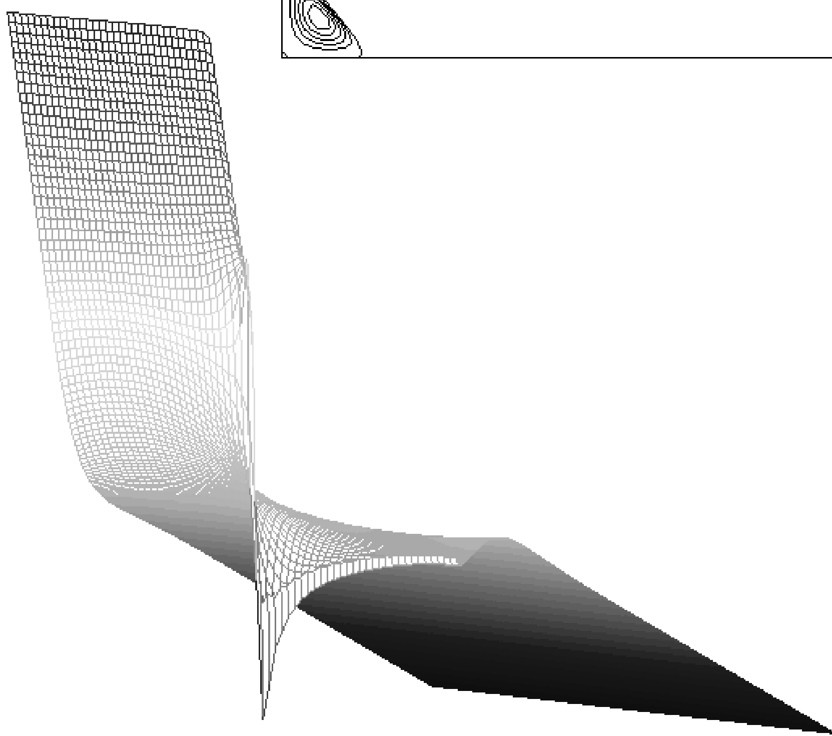
$$\begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial \vec{u}}{\partial t} \\ \frac{\partial p}{\partial t} \end{pmatrix} + \begin{pmatrix} N(\vec{u}) + \nu A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \vec{u} \\ p \end{pmatrix} = \begin{pmatrix} \vec{f} \\ 0 \end{pmatrix}$$

with associated matrices

$$N_{ij} = (\vec{u} \cdot \nabla \vec{\phi}_i, \vec{\phi}_j), \quad \text{convection}$$
$$A_{ij} = (\nabla \vec{\phi}_i, \nabla \vec{\phi}_j), \quad \text{diffusion}$$
$$B_{ij} = -(\nabla \cdot \vec{\phi}_j, \psi_i), \quad \text{divergence .}$$

Example: Flow over a Step



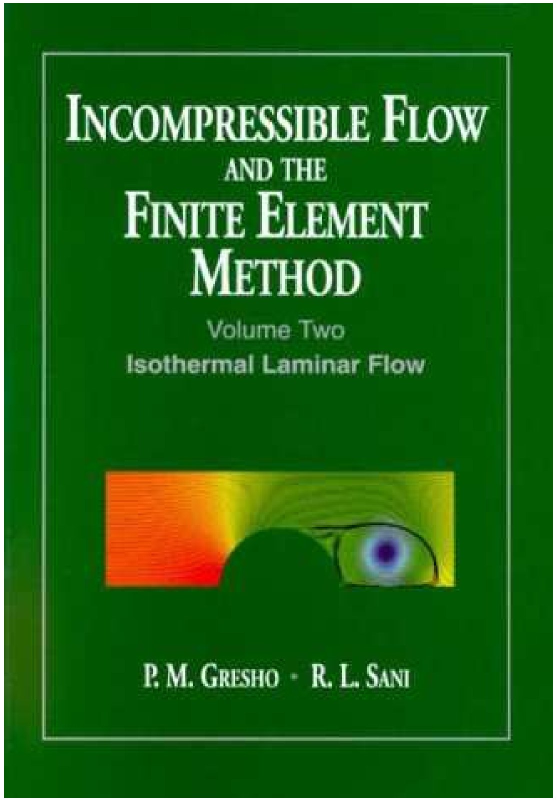


Outline

- PDEs
- Review : 1966 – 1999
- Update : 2000 – 2009

Outline

- PDEs
- Review : 1966 – 1999
  - 1967
  - 1982
  - 1991
  - 1999



Spatial Discretization— I

Suppose that $\Omega \subset \mathbb{R}^2$. Introducing $\vec{u} = (u_x, u_y)$ gives

$$\frac{\partial u_x}{\partial t} + \left(u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} \right) u_x - \nu \nabla^2 u_x + \frac{\partial p}{\partial x} = 0$$

$$\frac{\partial u_y}{\partial t} + \left(u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} \right) u_y - \nu \nabla^2 u_y + \frac{\partial p}{\partial y} = 0$$

$$- \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right) = 0$$

With a discrete analogue ..

Spatial Discretization— I

Suppose that $\Omega \subset \mathbb{R}^2$. Introducing $\vec{u} = (u_x, u_y)$ gives

$$\frac{\partial u_x}{\partial t} + \left(u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} \right) u_x - \nu \nabla^2 u_x + \frac{\partial p}{\partial x} = 0$$
$$\frac{\partial u_y}{\partial t} + \left(u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} \right) u_y - \nu \nabla^2 u_y + \frac{\partial p}{\partial y} = 0$$
$$- \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right) = 0$$

With a discrete analogue ..

$$\begin{pmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} \frac{\partial \alpha^x}{\partial t} \\ \frac{\partial \alpha^y}{\partial t} \\ \frac{\partial \alpha^p}{\partial t} \end{bmatrix} + \begin{pmatrix} F & 0 & B_x^T \\ 0 & F & B_y^T \\ B_x & B_y & 0 \end{pmatrix} \begin{bmatrix} \alpha^x \\ \alpha^y \\ \alpha^p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Spatial Discretization— II

The method-of-lines discretized system is a semi-explicit system of DAEs:

$$\begin{pmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} \frac{\partial \alpha^x}{\partial t} \\ \frac{\partial \alpha^y}{\partial t} \\ \frac{\partial \alpha^p}{\partial t} \end{bmatrix} + \begin{pmatrix} F & 0 & B_x^T \\ 0 & F & B_y^T \\ B_x & B_y & 0 \end{pmatrix} \begin{bmatrix} \alpha^x \\ \alpha^y \\ \alpha^p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- The DAEs have index equal to two
- The discrete problem is nonlinear $F := \nu A + N(\alpha)$

Spatial Discretization— II

The method-of-lines discretized system is a semi-explicit system of DAEs:

$$\begin{pmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} \frac{\partial \alpha^x}{\partial t} \\ \frac{\partial \alpha^y}{\partial t} \\ \frac{\partial \alpha^p}{\partial t} \end{bmatrix} + \begin{pmatrix} F & 0 & B_x^T \\ 0 & F & B_y^T \\ B_x & B_y & 0 \end{pmatrix} \begin{bmatrix} \alpha^x \\ \alpha^y \\ \alpha^p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- The DAEs have index equal to two
- The discrete problem is nonlinear $F := \nu A + N(\alpha)$
- To reduce the index we differentiate the constraint ...

Spatial Discretization— III

... to give an index one DAE system:

$$\begin{bmatrix} \frac{\partial \alpha^x}{\partial t} \\ \frac{\partial \alpha^y}{\partial t} \\ \mathbf{0} \end{bmatrix} + \begin{pmatrix} M^{-1}F & 0 & M^{-1}B_x^T \\ 0 & M^{-1}F & M^{-1}B_y^T \\ B_x M^{-1}F & B_y M^{-1}F & A_p \end{pmatrix} \begin{bmatrix} \alpha^x \\ \alpha^y \\ \alpha^p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- The matrix $A_p := B_x M^{-1} B_x^T + B_y M^{-1} B_y^T$ is the (consistent) Pressure Poisson matrix.
- Explicit approximation in time gives a decoupled formulation.
- Diagonally implicit approximation in time gives a segregated (SIMPLE-like) formulation.
- Implicit approximation in time does not look attractive!

$$\begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial \vec{u}}{\partial t} \\ \frac{\partial p}{\partial t} \end{pmatrix} + \begin{pmatrix} N(\vec{u}) + \nu A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \vec{u} \\ p \end{pmatrix} = \begin{pmatrix} \vec{f} \\ 0 \end{pmatrix}$$

- 1967

A simple splitting/projection approach — I

$$\begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial \vec{u}}{\partial t} \\ \frac{\partial p}{\partial t} \end{pmatrix} + \begin{pmatrix} F(\vec{u}) & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \vec{u} \\ p \end{pmatrix} = \begin{pmatrix} \vec{f} \\ 0 \end{pmatrix}$$

Given a time step dT and a parameter $0 < \gamma \leq 2$

Algorithm: Chorin, 1967; Temam, 1969

for $k = 0, 1, \ldots$

solve $M \frac{\partial \vec{u}_*}{\partial t} + F(\vec{u}_k)\, \vec{u}_* = \vec{f} - B^T p_k$

solve $B M^{-1} B^T \phi = B\, \vec{u}_*$

compute $\vec{u}_{k+1} = \vec{u}_* - M^{-1} B^T \phi$

compute $p_{k+1} = p_k + (\gamma / dT)\, \phi$

end

A simple splitting/projection approach — II

Key Question: Is the projection/splitting consistent?

A simple splitting/projection approach — II

Key Question: Is the projection/splitting consistent?
Practictioners say yes ...

- Chorin (1967), Temam(1969), Kim & Moin (1985), Van Kan (1986), Bell, Colella & Glaz (1989), Gresho & Chan (1984, 1990), Perot (1993), Turek (1997).

A simple splitting/projection approach — II

Key Question: Is the projection/splitting consistent?
Practictioners say yes ...

- Chorin (1967), Temam(1969), Kim & Moin (1985), Van Kan (1986), Bell, Colella & Glaz (1989), Gresho & Chan (1984, 1990), Perot (1993), Turek (1997).

Mathematicians say sometimes ...

- Guermond (1984), Rannacher (1992), E & Liu (1995, 1996), Shen (1992, 1996), Prohl (1997, 2007).

$$\begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial \vec{u}}{\partial t} \\ \frac{\partial p}{\partial t} \end{pmatrix} + \begin{pmatrix} N(\vec{u}) + \nu A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \vec{u} \\ p \end{pmatrix} = \begin{pmatrix} \vec{f} \\ 0 \end{pmatrix}$$

- 1967
- 1982

A Stokes splitting/projection approach — I

$$\begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial \vec{u}}{\partial t} \\ \frac{\partial p}{\partial t} \end{pmatrix} + \begin{pmatrix} N(\vec{u}) + \nu A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \vec{u} \\ p \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Given a time step dT, discretise the total derivative:

Algorithm: Pironneau, 1982; Hansbo, 1982

for $k = 0, 1, \ldots$

compute $X^m(\vec{x})$ the solution at $\tau_k = k\,dT$ of

$$\frac{dX}{d\tau} = \vec{u}_k(X, \tau), \quad X(\tau_{k+1}) = \vec{x};$$

interpolate $\vec{u}_k^* = \vec{u}_k(X^m(\vec{x}))$

solve
$$\begin{pmatrix} \frac{1}{dT} M + \nu A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \vec{u}_{k+1} \\ p_{k+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{dT} M\, \vec{u}_k^* \\ 0 \end{pmatrix}$$

end

A simple splitting/projection approach — II

Key[1] Question: Is the total derivative approximation stable numerically ?

A simple splitting/projection approach — II

Key[1] Question: Is the total derivative approximation stable numerically ?
Mathematicians say yes, but care is needed ...

- Hansbo (1982), Pironneau (1982), Morton, Priestley & Suli (1988,1989, 1994), Bermejo & Staniforth (1991, 1992).

A simple splitting/projection approach — II

Key^1 Question: Is the total derivative approximation stable numerically ?
Mathematicians say yes, but care is needed ...

- Hansbo (1982), Pironneau (1982), Morton, Priestley & Suli (1988,1989, 1994), Bermejo & Staniforth (1991, 1992).

Key^2 Question: Adaptive time stepping ?

$$\begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial \vec{u}}{\partial t} \\ \frac{\partial p}{\partial t} \end{pmatrix} + \begin{pmatrix} N(\vec{u}) + \nu A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \vec{u} \\ p \end{pmatrix} = \begin{pmatrix} \vec{f} \\ 0 \end{pmatrix}$$

- 1967
- 1982
- 1991

A clever splitting/projection approach — I

$$\begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial \vec{u}}{\partial t} \\ \frac{\partial p}{\partial t} \end{pmatrix} + \begin{pmatrix} N(\vec{u}) + \nu A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \vec{u} \\ p \end{pmatrix} = \begin{pmatrix} \vec{f} \\ 0 \end{pmatrix}$$

Given dT and parameters $\theta = 1 - 1/\sqrt{2}$, $\alpha = \frac{1-2\theta}{1-\theta}, \beta = 1 - \alpha$.

Algorithm: Glowinski, 1986, 1991

for $k = 0, 1, \ldots$

solve $$\begin{pmatrix} \frac{1}{\theta dT} M + \alpha\nu A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \vec{u}_{k+\theta} \\ p_{k+\theta} \end{pmatrix} = \begin{pmatrix} f_k \\ 0 \end{pmatrix}$$

solve $$\left(\frac{1}{(1-2\theta)dT} M + N(\vec{u}_{k+1-\theta}) + \beta\nu A\right) \vec{u}_{k+1-\theta} = \vec{f}_{k+\theta}$$

solve $$\begin{pmatrix} \frac{1}{\theta dT} M + \alpha\nu A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \vec{u}_{k+1} \\ p_{k+1} \end{pmatrix} = \begin{pmatrix} f_{k+1-\theta} \\ 0 \end{pmatrix}$$

end

A clever splitting/projection approach — II

Key[1] Question: Is the algorithm over-complicated?

A clever splitting/projection approach — II

Key[1] Question: Is the algorithm over-complicated?
Mathematicians say no, ...

- Glowinski & Dean (1984, 1993) , Bristeau et al. (1985), Rannacher (1989, 1990), Turek (1996), Smith & Silvester (1997).

A clever splitting/projection approach — II

Key^1 Question: Is the algorithm over-complicated?
Mathematicians say no, ...

- Glowinski & Dean (1984, 1993) , Bristeau et al. (1985), Rannacher (1989, 1990), Turek (1996), Smith & Silvester (1997).

Key^2 Question: Adaptive time stepping ?

- Update : 2000 – 2009

- Philip Gresho & David Griffiths & David Silvester Adaptive time-stepping for incompressible flow; part I: scalar advection-diffusion, SIAM J. Scientific Computing, 30: 2018–2054, 2008.
- David Kay & Philip Gresho & David Griffiths & David Silvester Adaptive time-stepping for incompressible flow; part II: Navier-Stokes equations. MIMS Eprint 2008.61.

“Smart Integrator” (SI) definition

- Optimal time-stepping: time-steps automatically chosen to “follow the physics”.
- Black-box implementation: few parameters that have to be estimated a priori.

“Smart Integrator” (SI) definition

- Optimal time-stepping: time-steps automatically chosen to “follow the physics”.
- Black-box implementation: few parameters that have to be estimated a priori.
- Solver efficiency: the linear solver convergence rate is robust with respect to the mesh size h and the Reynolds number $1/\nu$.

Trapezoidal Rule (TR) time discretization

We subdivide $[0,T]$ into time levels $\{t_i\}_{i=1}^N$. Given $(\vec{u}^n, p^n)$ at time level t_n, $k_{n+1} := t_{n+1} - t_n$, compute $(\vec{u}^{n+1}, p^{n+1})$ via

$$\frac{2}{k_{n+1}}\vec{u}^{n+1} + \vec{w}^{n+1}\cdot\nabla\vec{u}^{n+1} - \nu\nabla^2\vec{u}^{n+1} + \nabla p^{n+1} = \vec{f}^{n+1}$$

$$-\nabla\cdot\vec{u}^{n+1} = 0 \qquad \text{in } \Omega$$

$$\vec{u}^{n+1} = \vec{g}^{n+1} \qquad \text{on } \Gamma_D$$

$$\nu\nabla\vec{u}^{n+1}\cdot\vec{n} - p^{n+1}\vec{n} = \vec{0} \qquad \text{on } \Gamma_N$$

with second-order linearization

$$\vec{f}^{n+1} = \frac{2}{k_{n+1}}\vec{u}^n + \nu\nabla^2\vec{u}^n - \vec{u}^n\cdot\nabla\vec{u}^n - \nabla p^n$$

$$\vec{w}^{n+1} = \left(1 + \frac{k_{n+1}}{k_n}\right)\vec{u}^n - \frac{k_{n+1}}{k_n}\vec{u}^{n-1}$$

Saddle-point system

In $\mathbb{R}^2$ the discretized Oseen system (*) is:

$$\begin{pmatrix} F^{n+1} & 0 & B_x^T \\ 0 & F^{n+1} & B_y^T \\ B_x & B_y & 0 \end{pmatrix} \begin{bmatrix} \alpha^{x,n+1} \\ \alpha^{y,n+1} \\ \alpha^{p,n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{f}^{x,n+1} \\ \mathbf{f}^{y,n+1} \\ \mathbf{f}^{p,n+1} \end{bmatrix}$$

- $F^{n+1} := \frac{2}{k_{n+1}} M + \nu A + N(\vec{w}_h^{n+1})$
- The vector f is constructed from the boundary data $\vec{g}^{n+1}$, the computed velocity $\vec{u}_h^n$ at the previous time level and the acceleration $\frac{\partial \vec{u}_h^n}{\partial t}$
- The system can be efficiently solved using “appropriately” preconditioned GMRES...

Preconditioned system

$$\begin{pmatrix} F & B^T \\ B & 0 \end{pmatrix} \mathcal{P}^{-1} \quad \mathcal{P} \begin{pmatrix} \alpha^u \\ \alpha^p \end{pmatrix} = \begin{pmatrix} \mathbf{f}^u \\ \mathbf{f}^p \end{pmatrix}$$

A perfect preconditioner is given by

$$\begin{pmatrix} F & B^T \\ B & 0 \end{pmatrix} \underbrace{\begin{pmatrix} F^{-1} & F^{-1}B^T S^{-1} \\ 0 & -S^{-1} \end{pmatrix}}_{\mathcal{P}^{-1}} = \begin{pmatrix} I & 0 \\ BF^{-1} & I \end{pmatrix}$$

with $F = \frac{2}{k_{n+1}} M + \nu A + N$ and $S = BF^{-1}B^T$.

For an efficient preconditioner we need to construct a sparse approximation to the “exact” Schur complement

$$S^{-1} = (BF^{-1}B^T)^{-1}$$

See Chapter 8 of

- Howard Elman & David Silvester & Andrew Wathen
  Finite Elements and Fast Iterative Solvers: with applications in incompressible fluid dynamics
  Oxford University Press, 2005.

Two possible constructions ...

Schur complement approximation – I

Introducing the diagonal of the velocity mass matrix

$$M_* \sim M_{ij} = (\vec{\phi}_i, \vec{\phi}_j),$$

gives the "least-squares commutator preconditioner":

$$(BF^{-1}B^T)^{-1} \approx (\underbrace{BM_*^{-1}B^T}_{\text{AMG}})^{-1}(BM_*^{-1}FM_*^{-1}B^T)(\underbrace{BM_*^{-1}B^T}_{\text{AMG}})^{-1}$$

Schur complement approximation – II

Introducing associated pressure matrices

$$
\begin{aligned}
M_p &\sim (\nabla\psi_i, \nabla\psi_j), && \text{mass}\\
A_p &\sim (\nabla\psi_i, \nabla\psi_j), && \text{diffusion}\\
N_p &\sim (\,\vec{w}_h \cdot \nabla\psi_i, \psi_j), && \text{convection}\\
F_p &= \frac{2}{k_{n+1}} M_p + \nu A_p + N_p, && \text{convection-diffusion}
\end{aligned}
$$

gives the “pressure convection-diffusion preconditioner”:

$$
(BF^{-1}B^T)^{-1} \approx M_p^{-1} F_p \underbrace{A_p^{-1}}_{\mathrm{AMG}}
$$

Adaptive Time Stepping AB2–TR

Consider the simple ODE $\dot{u} = f(u)$
Manipulating the truncation error terms for TR and AB2 gives the estimate

$$T_n = \frac{u_{n+1} - u^*_{n+1}}{3(1 + \frac{k_n}{k_{n+1}})}$$

Given some user-prescribed error tolerance `tol`, the new time step is selected to be the biggest possible such that $\|T_{n+1}\| \leq \texttt{tol} \times u_{\max}$. This criterion leads to

$$k_{n+2} := k_{n+1} \left(\frac{\texttt{tol} \times u_{\max}}{\|T_n\|} \right)^{1/3}$$

Adaptive Time Stepping AB2–TR

Consider the simple ODE $\dot{u} = f(u)$
Manipulating the truncation error terms for TR and AB2 gives the estimate

$$T_n = \frac{u_{n+1} - u^*_{n+1}}{3(1 + \frac{k_n}{k_{n+1}})}$$

Given some user-prescribed error tolerance `tol`, the new time step is selected to be the biggest possible such that $\|T_{n+1}\| \leq \texttt{tol} \times u_{\max}$. This criterion leads to

$$k_{n+2} := k_{n+1} \left(\frac{\texttt{tol} \times u_{\max}}{\|T_n\|} \right)^{1/3}$$

But look out for “ringing” ...

Stabilized AB2–TR

To address the instability issues:

- We rewrite the AB2–TR algorithm to compute updates v_n and w_n scaled by the time-step:

$$u_{n+1} - u_n = \tfrac{1}{2} k_{n+1} v_n; \quad u^*_{n+1} - u^*_n = k_{n+1} w_n.$$

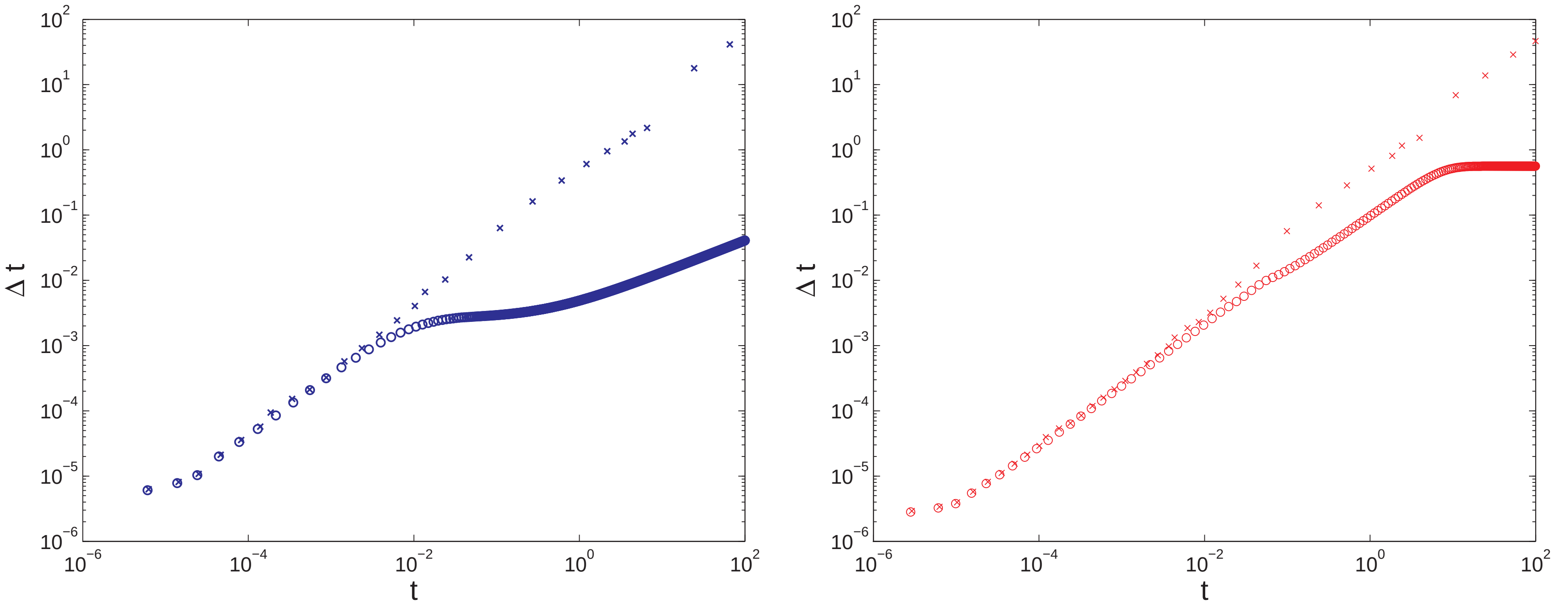
- We perform time-step averaging every n^* steps:

$$u_n := \tfrac{1}{2}(u_n + u_{n-1}); \quad u_{n+1} := u_n + \tfrac{1}{4} k_{n+1} v_n; \quad \dot{u}_{n+1} := \tfrac{1}{2} v_n.$$

Contrast this with the standard acceleration obtained by "inverting" the TR formula:

$$\dot{u}_{n+1} = \frac{2}{k_{n+1}} \left(u_{n+1} - u_n\right) - \dot{u}_n = v_n - \dot{u}_n$$

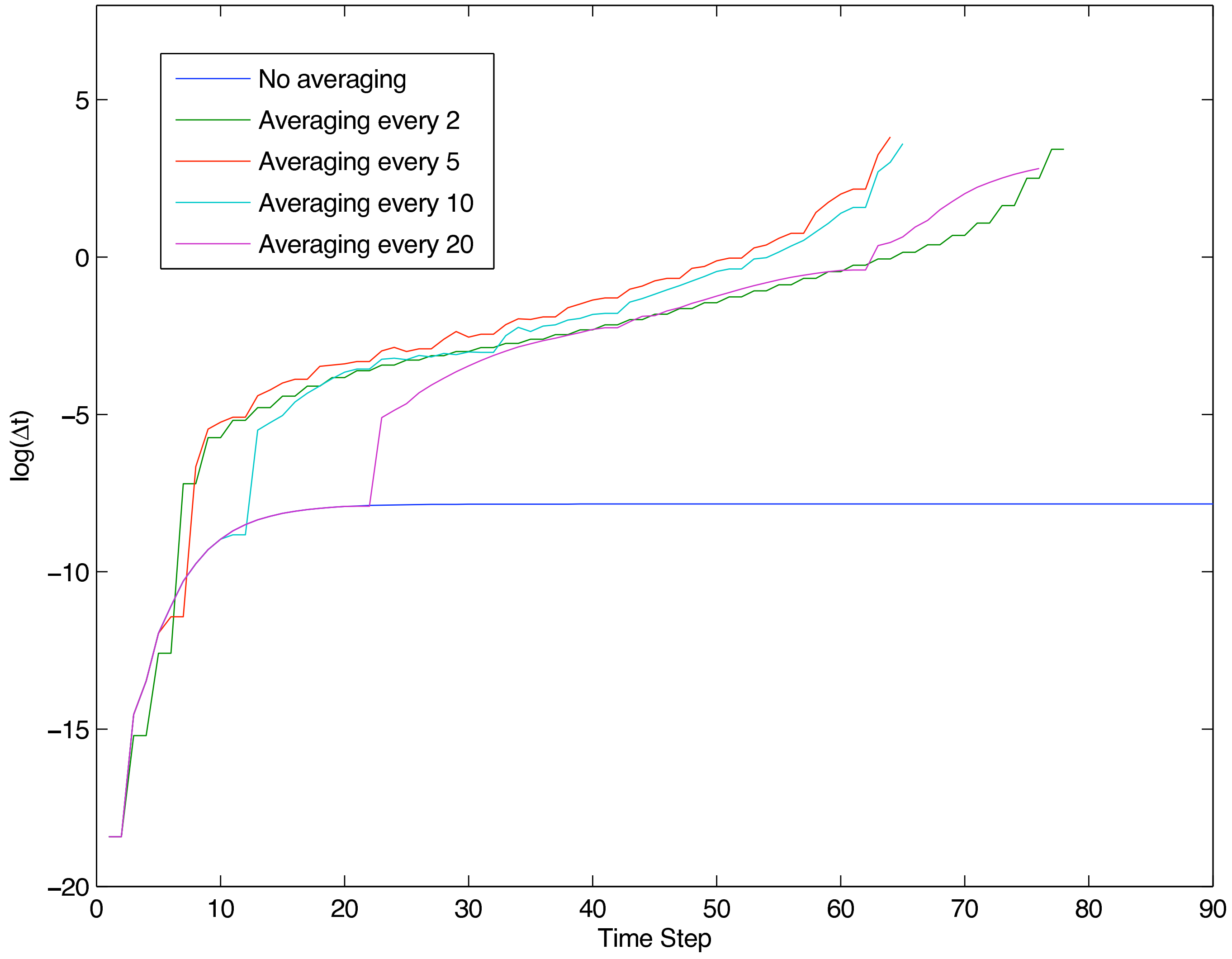
Stabilized AB2–TR



Advection-Diffusion of step profile on Shishkin grid.

$\texttt{tol} = 10^{-3}$ $\texttt{tol} = 10^{-4}$

Stabilized AB2–TR



“Spin up” driven cavity flow with $\nu = 1/100$.

Adaptive Time-Stepping Algorithm I

- The following parameters must be specified:

time accuracy tolerance	tol	(10^{-4})
GMRES tolerance	`itol`	(10^{-6})
GMRES iteration limit	`maxit`	(50)

Adaptive Time-Stepping Algorithm I

- The following parameters must be specified:

time accuracy tolerance	tol	(10^{-4})
GMRES tolerance	`itol`	(10^{-6})
GMRES iteration limit	`maxit`	(50)

- Starting from rest, $\vec{u}^0 = \vec{0}$, and given a steady state boundary condition $\vec{u}(\vec{x}, t) = \vec{g}$, we model the impulse with a time-dependent boundary condition:

$$\vec{u}(\vec{x}, t) = \vec{g}(1 - e^{-5t}) \quad \text{on } \Gamma_D \times [0, T].$$

Adaptive Time-Stepping Algorithm I

- The following parameters must be specified:

time accuracy tolerance	tol	(10^{-4})
GMRES tolerance	`itol`	(10^{-6})
GMRES iteration limit	`maxit`	(50)

- Starting from rest, $\vec{u}^0 = \vec{0}$, and given a steady state boundary condition $\vec{u}(\vec{x}, t) = \vec{g}$, we model the impulse with a time-dependent boundary condition:

$$\vec{u}(\vec{x}, t) = \vec{g}(1 - e^{-5t}) \quad \text{on } \Gamma_D \times [0, T].$$

- We specify the frequency of averaging, typically $n_* = 10$. We also choose a very small initial timestep, typically, $k_1 = 10^{-8}$.

Adaptive Time-Stepping Algorithm II

- Setup the Oseen System ($*$) and compute $[\alpha^{x,n+1}, \alpha^{y,n+1}]$ using GMRES(`maxit`, `itol`).

Adaptive Time-Stepping Algorithm II

- Setup the Oseen System ($*$) and compute $[\alpha^{x,n+1}, \alpha^{y,n+1}]$ using GMRES(`maxit`, `itol`).
- Compute the LTE estimate $\mathbf{e}^{v,n+1}$

Adaptive Time-Stepping Algorithm II

- Setup the Oseen System $(*)$ and compute $[\alpha^{x,n+1}, \alpha^{y,n+1}]$ using GMRES(`maxit`, `itol`).

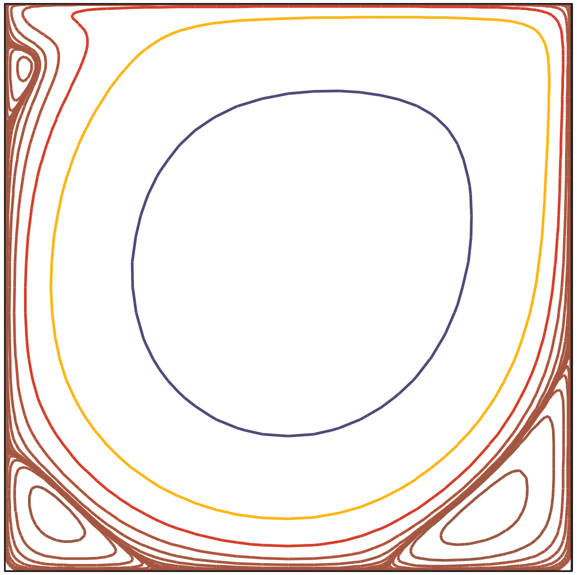
- Compute the LTE estimate $\mathbf{e}^{v,n+1}$

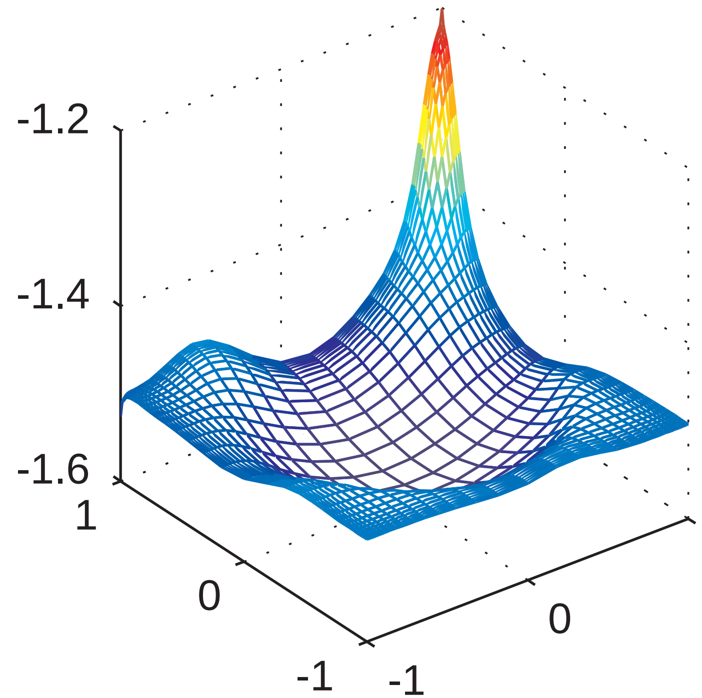
- If $\|\mathbf{e}^{v,n+1}\| > (1/0.7)^3$tol, we reject the current time step, and repeat the old time step with $k_{n+1} = k_{n+1}(\frac{\text{tol}}{\|\mathbf{e}^{v,n+1}\|})^{1/3}$.

Adaptive Time-Stepping Algorithm II

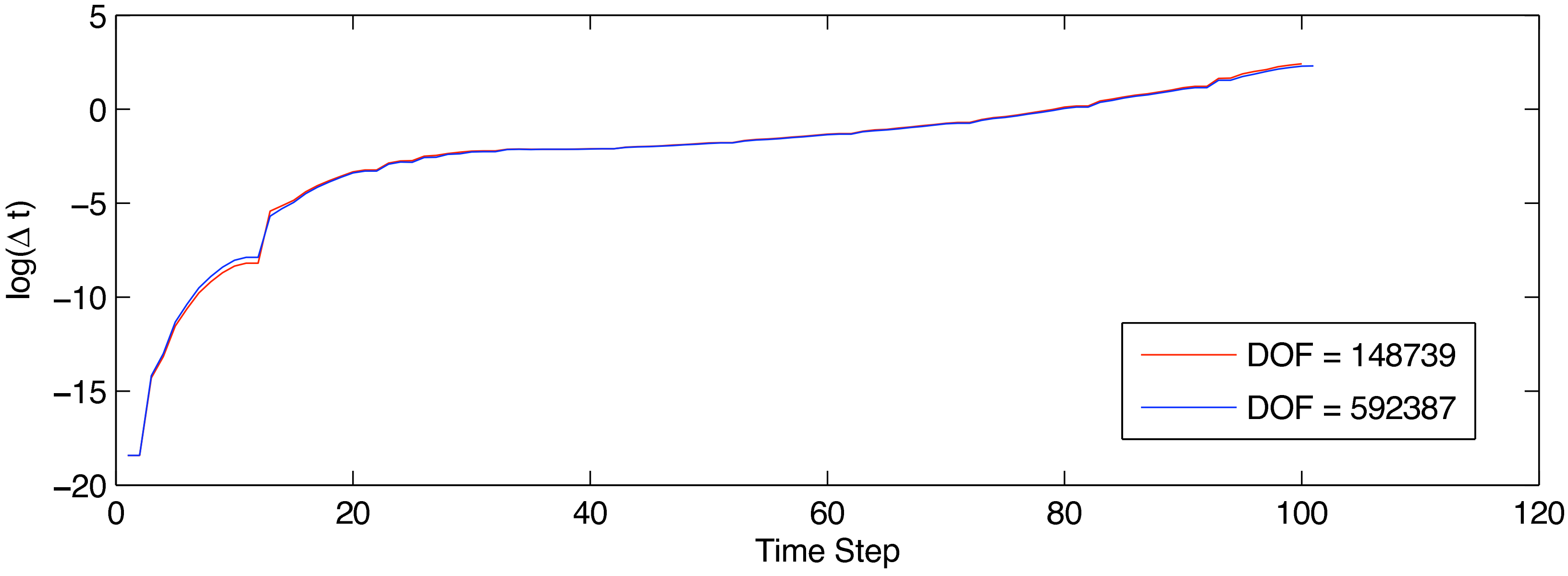
- Setup the Oseen System ($*$) and compute $[\alpha^{x,n+1}, \alpha^{y,n+1}]$ using GMRES(`maxit`, `itol`).
- Compute the LTE estimate $\mathbf{e}^{v,n+1}$
- If $\|\mathbf{e}^{v,n+1}\| > (1/0.7)^3$tol, we reject the current time step, and repeat the old time step with $k_{n+1} = k_{n+1}(\frac{\text{tol}}{\|\mathbf{e}^{v,n+1}\|})^{1/3}$.
- Otherwise, accept the step and continue with $n = n+1$ and k_{n+2} based on the LTE estimate and the accuracy tolerance tol.

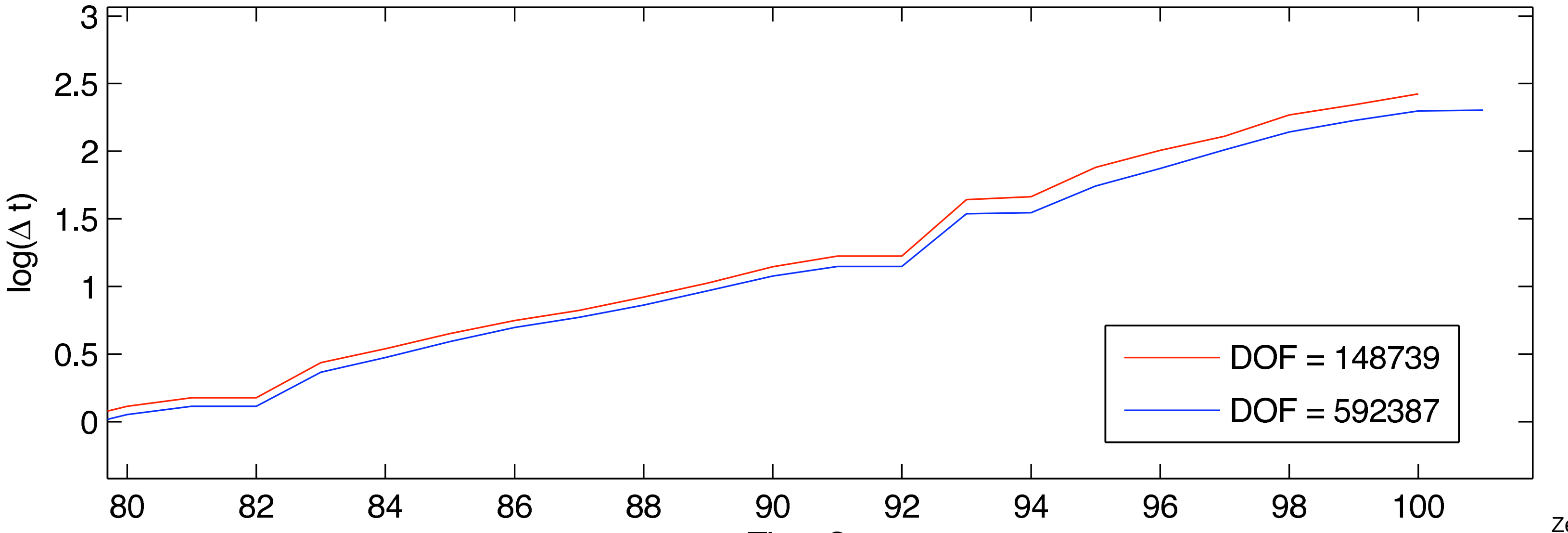
Example Flow Problem – I ($\nu = 1/1000$)



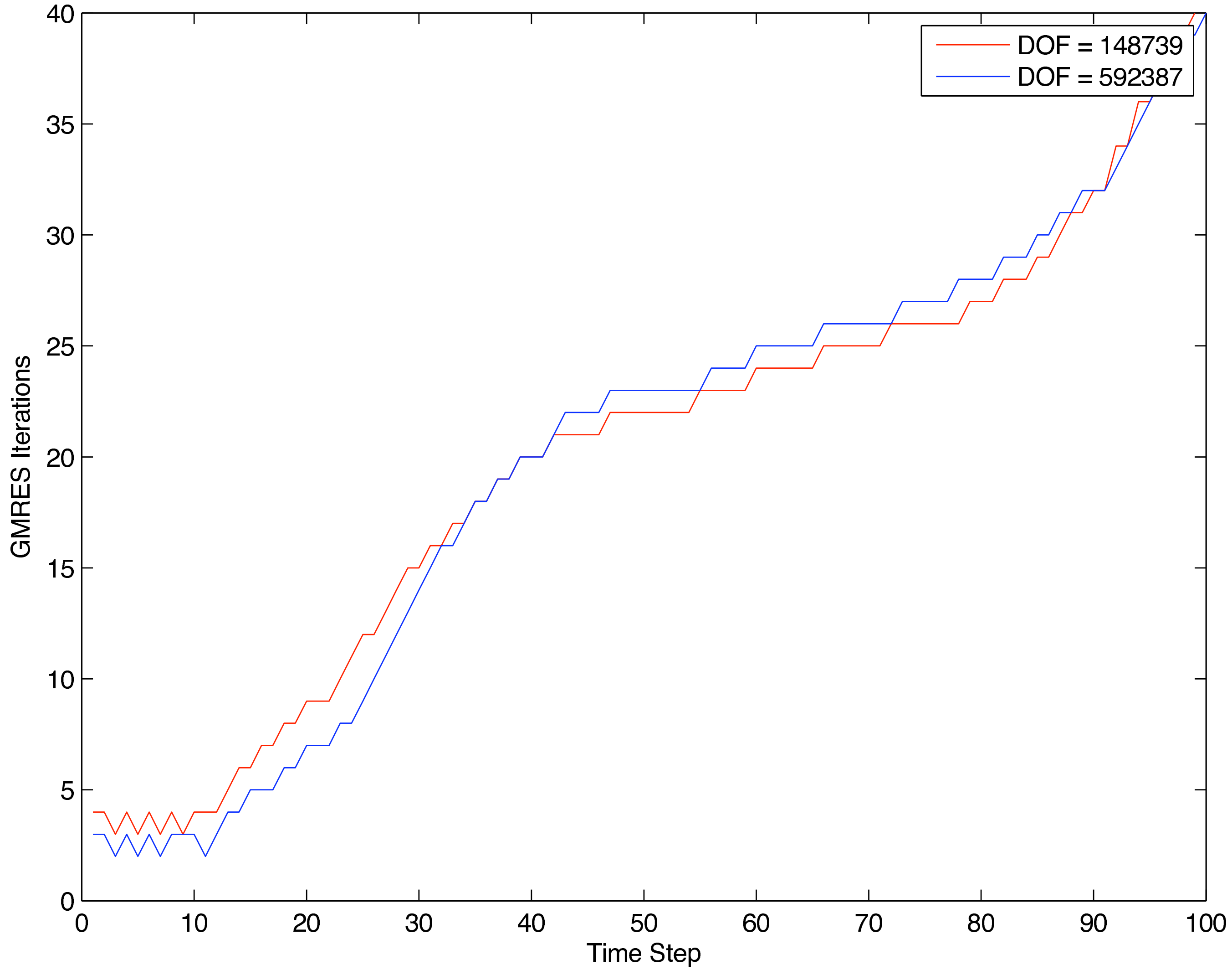


Time step evolution

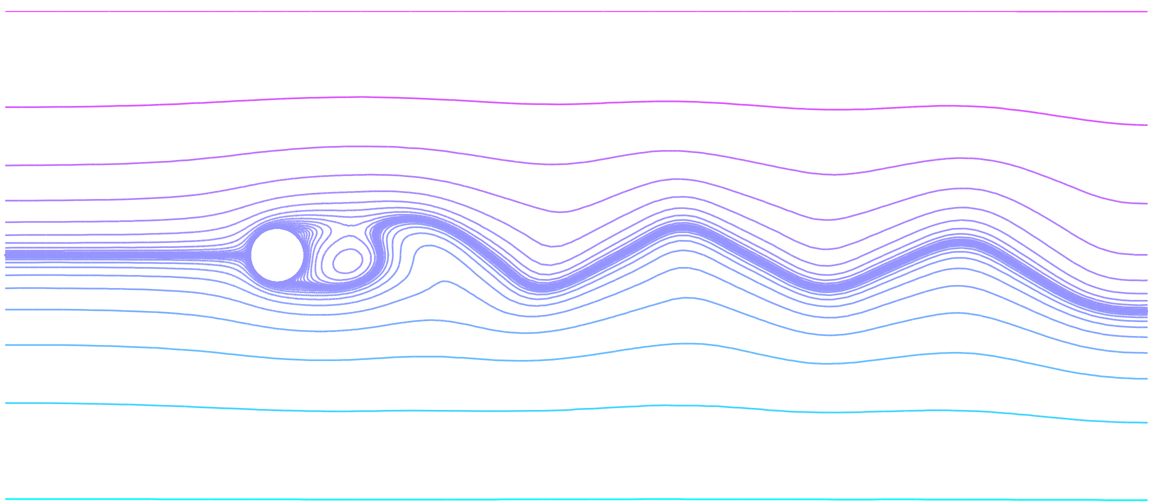


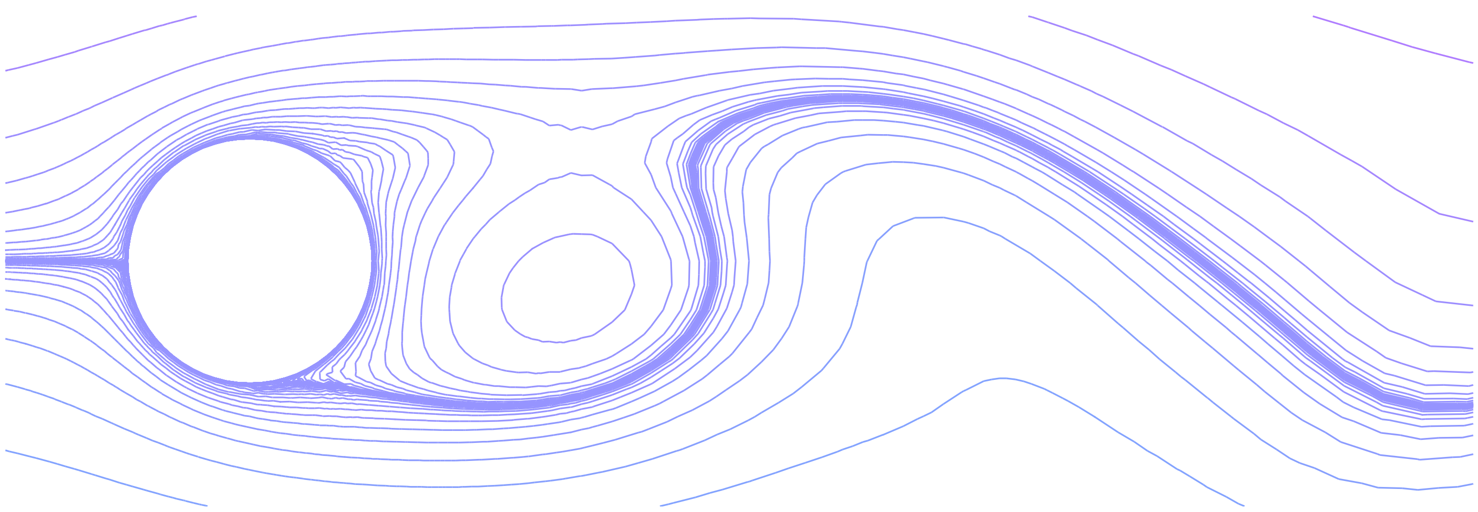


Linear solver performance

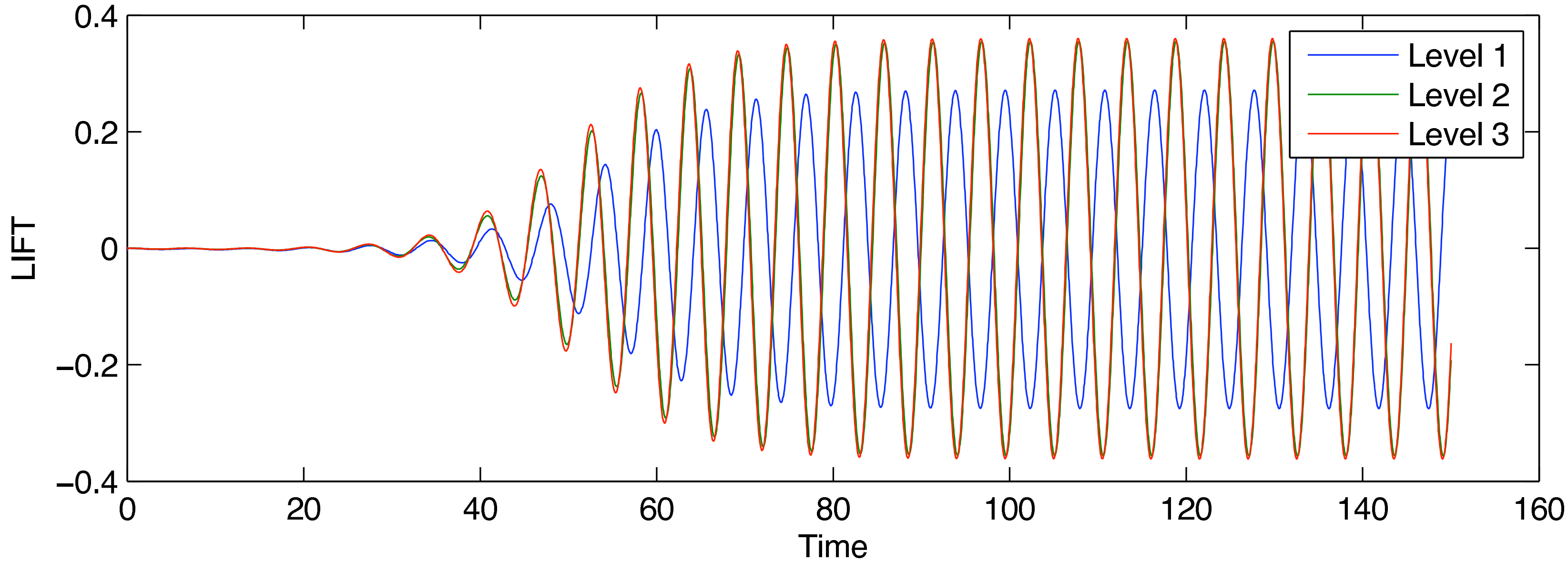


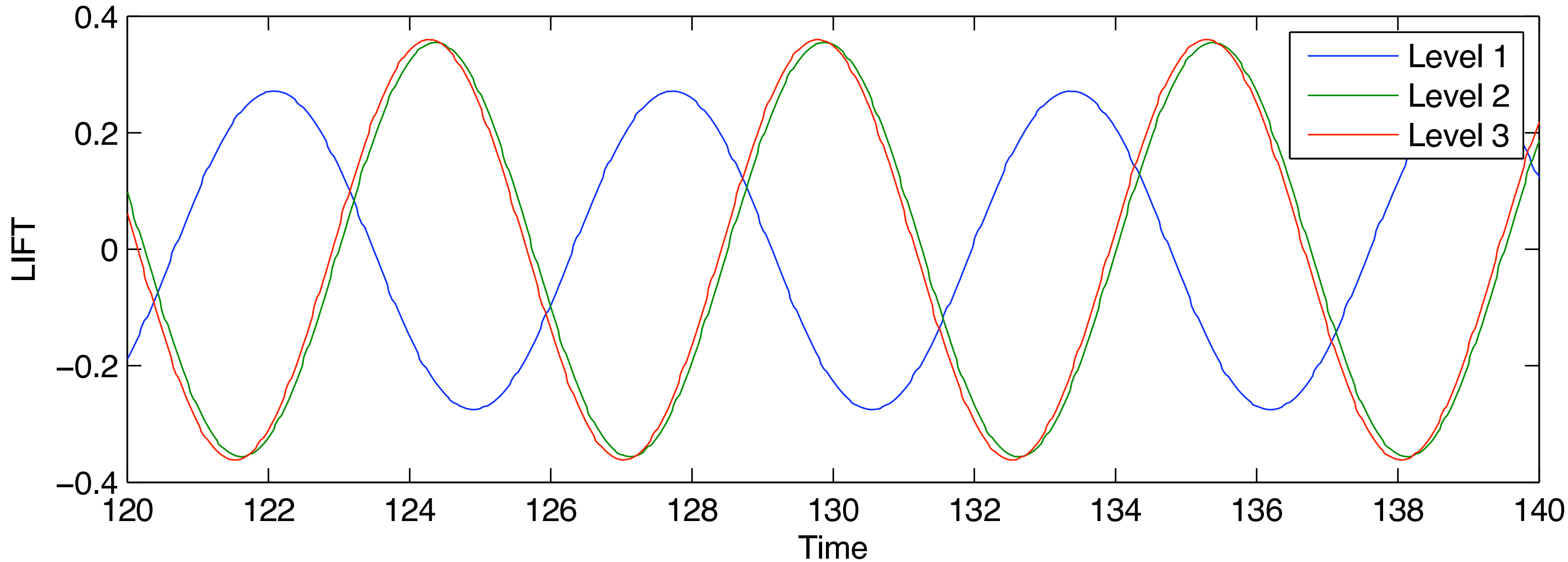
Example Flow Problem – II ($\nu = 1/100$)



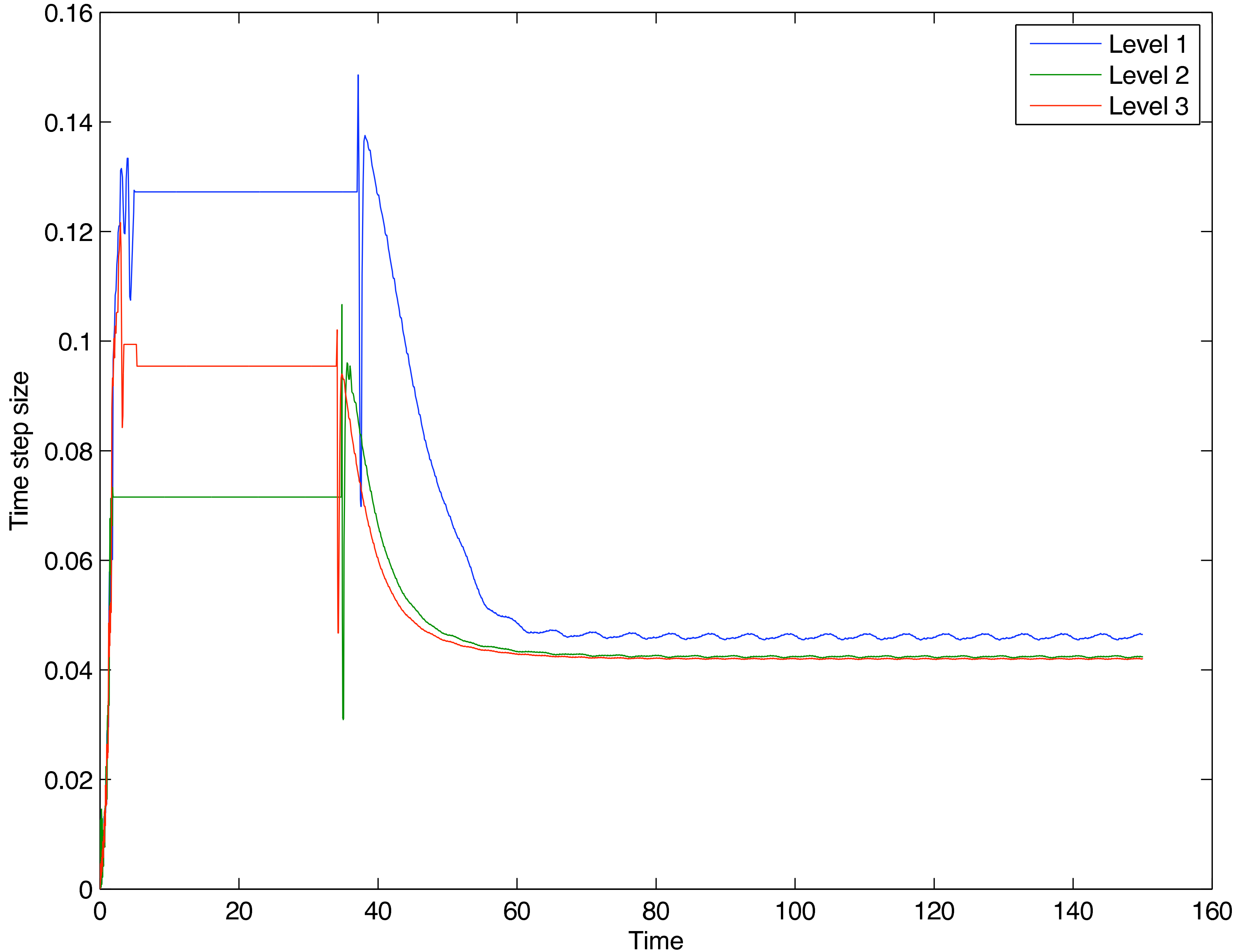


Lift Coefficient





Time step evolution



Bouyancy driven flow

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} - \nu \nabla^2 \vec{u} + \nabla p = \vec{j} T \quad \text{in } \mathcal{W} \equiv \Omega \times (0, T)$$

$$\nabla \cdot \vec{u} = 0 \qquad \text{in } \mathcal{W}$$

$$\frac{\partial T}{\partial t} + \vec{u} \cdot \nabla T - \nu \nabla^2 T = 0 \qquad \text{in } \mathcal{W}$$

Boundary and Initial conditions

$$\vec{u} = \vec{0} \quad \text{on } \Gamma \times [0, T]; \qquad \vec{u}(\vec{x}, 0) = \vec{0} \quad \text{in } \Omega.$$

$$T = T_g \quad \text{on } \Gamma_D \times [0, T]; \qquad \nu \nabla T \cdot \vec{n} = 0 \quad \text{on } \Gamma_N \times [0, T];$$

$$T(\vec{x}, 0) = T_0(\vec{x}) \quad \text{in } \Omega.$$

Finite element matrix formulation

Introducing the basis sets

$$\mathbf{X}_h = \text{span}\{\vec{\phi}_i\}_{i=1}^{n_u}, \quad \text{Velocity basis functions;}$$
$$M_h = \text{span}\{\psi_j\}_{j=1}^{n_p}, \quad \text{Pressure basis functions.}$$
$$T_h = \text{span}\{\phi_k\}_{k=1}^{n_T}, \quad \text{Temperature basis functions;}$$

gives the method-of-lines discretized system:

$$\begin{pmatrix} M & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & M \end{pmatrix} \begin{pmatrix} \frac{\partial \vec{u}}{\partial t} \\ \frac{\partial p}{\partial t} \\ \frac{\partial T}{\partial t} \end{pmatrix} + \begin{pmatrix} F & B^T & -\overset{\circ}{\overline{M}} \\ B & 0 & 0 \\ 0 & 0 & F \end{pmatrix} \begin{pmatrix} \vec{u} \\ p \\ T \end{pmatrix} = \begin{pmatrix} \vec{0} \\ 0 \\ g \end{pmatrix}$$

with a (vertical–) mass matrix:

$$(\overset{\circ}{\overline{M}})_{ij} = ([0, \phi_i], \phi_j)$$

Preconditioning strategy

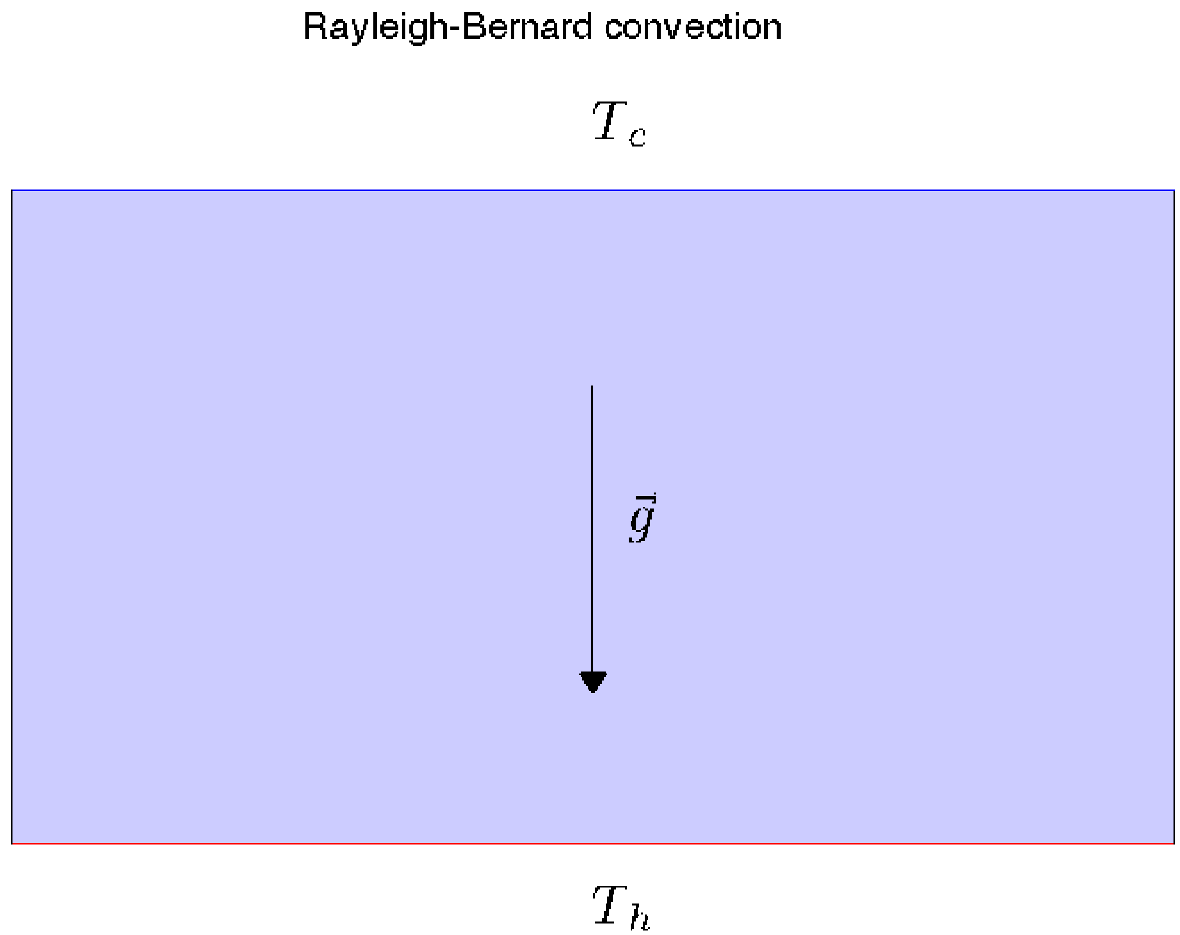
$$\begin{pmatrix} F & B^T & -\overset{\circ}{M} \\ B & 0 & 0 \\ 0 & 0 & F \end{pmatrix} \mathcal{P}^{-1} \quad \mathcal{P} \begin{pmatrix} \alpha^u \\ \alpha^p \\ \alpha^T \end{pmatrix} = \begin{pmatrix} \mathbf{f}^u \\ \mathbf{f}^p \\ \mathbf{f}^T \end{pmatrix}$$

Given $S = BF^{-1}B^T$, a perfect preconditioner is given by

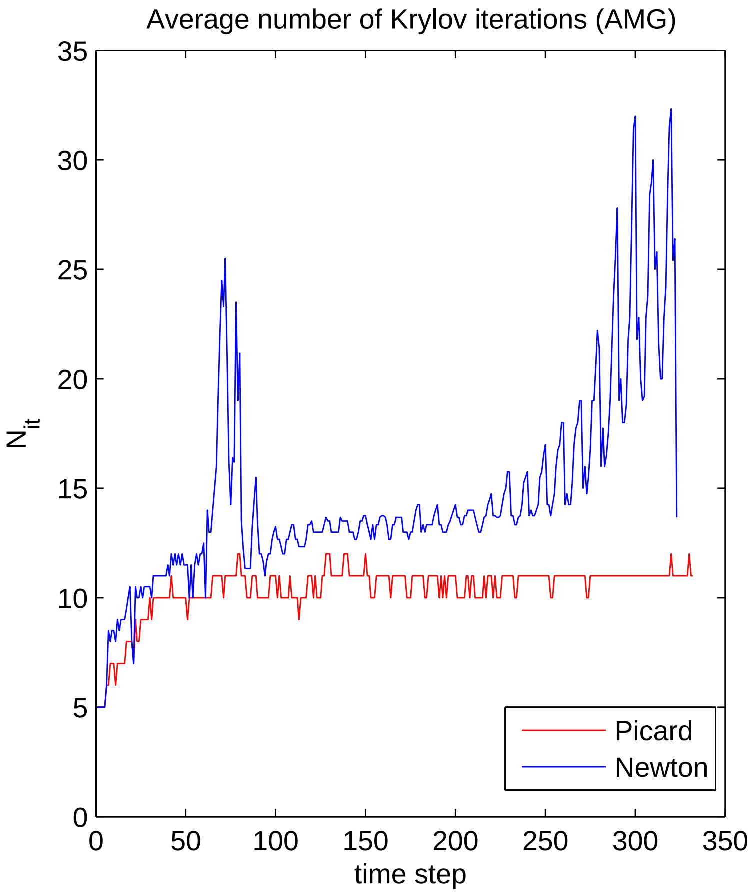
$$\begin{pmatrix} F & B^T & -\overset{\circ}{M} \\ B & 0 & 0 \\ 0 & 0 & F \end{pmatrix} \underbrace{\begin{pmatrix} F^{-1} & F^{-1}B^T S^{-1} & F^{-1}\overset{\circ}{M}F^{-1} \\ 0 & -S^{-1} & 0 \\ 0 & 0 & F^{-1} \end{pmatrix}}_{\mathcal{P}^{-1}}$$

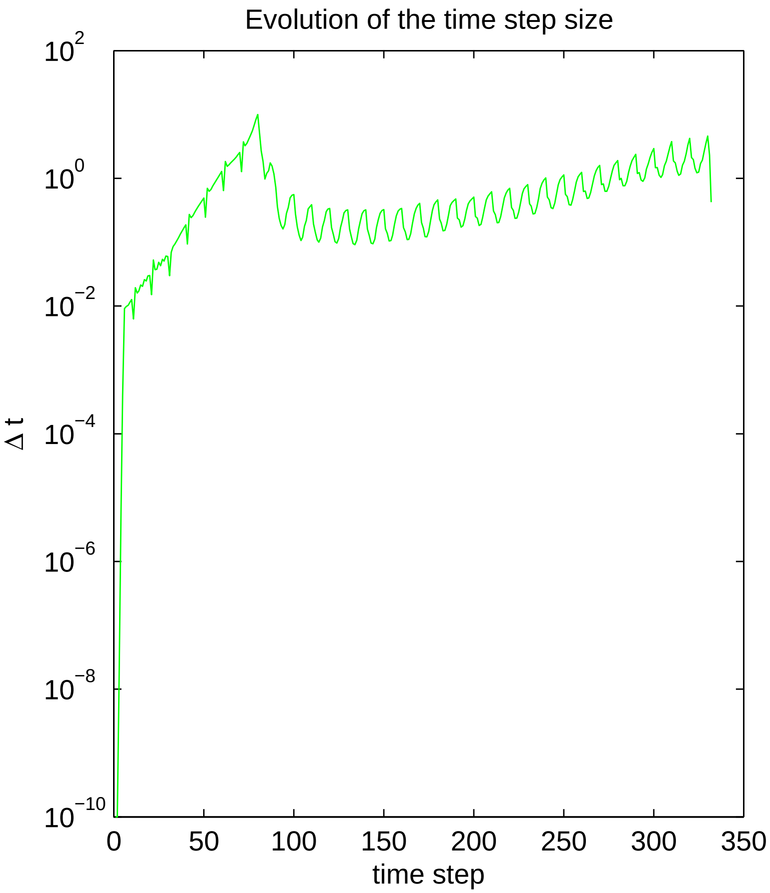
$$= \begin{pmatrix} I & 0 & 0 \\ BF^{-1} & I & BF^{-1}\overset{\circ}{M}F^{-1} \\ 0 & 0 & I \end{pmatrix}$$

Example: Natural Convection in 1:16 cavity



... with efficient linear algebra





What have we achieved?

- Black-box implementation: few parameters that have to be estimated a priori.
- Optimal complexity: essentially $O(n)$ flops per iteration, where n is dimension of the discrete system.
- Efficient linear algebra: convergence rate is (essentially) independent of h. Given an appropriate time accuracy tolerance convergence is also robust with respect to ν

References

Semi-Implicit, Fractional-Step, Pressure Projection — I

- A.J. Chorin
  A numerical method for solving incompressible viscous flow problems, J. Comput. Phys., 2:12–26, 1967.
- J. Kim & P. Moin
  Application of a fractional–step method to incompressible Navier–Stokes equations,
  J. Comput. Phys., 59:308–323, 1985.
- J. Van Kan
  A second–order accurate pressure–correction scheme for viscous incompressible flow,
  SIAM J. Sci. Stat. Comput., 7(3):870–891, 1986.
- J.B. Bell, P. Colella & H.M. Glaz
  A second–order projection method for the incompressible Navier–Stokes equations,
  J. Comput. Phys., 85:(2)257–283, 1989.

Semi-Implicit, Fractional-Step, Pressure Projection — II

- P.M. Gresho, S.T. Chan, R.L. Lee & C.D. Upson
  A modified finite element method for solving the time–dependent, incompressible Navier–Stokes equations. Part 1: Theory,
  Int. J. Numer. Meth. Fluids, 4:557–598, 1984.
- P.M. Gresho & S.T. Chan
  On the theory of semi–implicit projection methods for viscous incompressible flow and its implementation via a finite element method that also introduces a nearly consistent mass matrix. Part 2: Implementation,
  Int. J. Numer. Meth. Fluids, 11(5):621-660, 1990.
- J.B. Perot
  An analysis of the fractional step method,
  J. Comput. Phys., 108:51–58, 1993.

Semi-Implicit, Fractional-Step, Pressure Projection — III

- J.–L. Guermond
  Sur l'approximation des équations de Navier–Stokes instationnaires par une méthode de projection, C.R. Acad. Sci. Paris, 319:887–892, 1994. Serie I.
- R. Rannacher
  The Navier–Stokes Equations II: Theory and Numerical Methods, Springer–Verlag, Berlin, Germany, 1992. Chap.On Chorin's projection method for the incompressible Navier–Stokes equations; pp. 167–183; Lecture Notes in Mathematics, Vol. 1530.

Semi-Implicit, Fractional-Step, Pressure Projection — IV

- W. E & J.–G. Liu
  Projection method I: Convergence and numerical boundary layers,
  SIAM J. Numer. Anal., 32(4):1017–1057, 1995.
- W. E & J.–G. Liu
  Vorticity boundary conditions and related issues for finite difference schemes,
  J. Comput. Phys., 124:368–382, 1996.

Semi-Implicit, Fractional-Step, Pressure Projection — V

- J. Shen
  On error estimates of some higher order projection and penalty–projection methods for Navier–Stokes equations, Numer. Math., 62:49–73, 1992.
- J. Shen
  On error estimates of the projection methods for the Navier–Stokes equations: Second–order schemes, Math. Comput., 65, 215:1039–1065, 1996.
- A. Prohl
  Analysis of Chorin's projection method for solving the incompressible Navier–Stokes equations, Universität Heidelberg, Institut für Angewandte Mathematik, INF 294, D–69120 Heidelberg, Germany, 1996.

Transport Diffusion, Lagrange-Galerkin, Backward Method of Characteristics — I

- P. Hansbo
  The characteristic streamline diffusion method for the time–dependent incompressible Navier–Stokes equations, Comput. Meth. Appl. Mech. Eng., 99:171–186, 1992.
- O. Pironneau
  On the transport–diffusion algorithm and its application to the Navier–Stokes equations, Numer. Math., 38:309–332, 1982.
- K.W. Morton, A. Priestley & E. Süli
  Stability of the Lagrange–Galerkin method with non–exact integration,
  Math. Model. Numer. Anal., 22:(4)625–653, 1988.

Transport Diffusion, Lagrange-Galerkin, Backward Method of Characteristics — II

- A. Priestley
  Exact projections and the Lagrange–Galerkin method: A realistic alternative to quadrature, J. Comput. Phys., 112:316–333, 1994
- R. Bermejo
  Analysis of an algorithm for the Galerkin–characteristic method, Numer. Math., 60:163–194, 1991.

Fractional Step, le Θ scheme — I

- M.O. Bristeau, R. Glowinski, B. Mantel, J. Periaux & P. Perrier, Numerical methods for incompressible and compressible Navier–Stokes problems, Finite Elements in Fluids 6 (R.H. Gallagher, G. Carey, J.T. Oden & O.C. Zienkiewicz, eds), Chichester: Wiley, 1985. pp. 1–40.
- E.J. Dean & R. Glowinski
  On some finite element methods for the numerical simulation of incompressible viscous flow,
  Incompressible Computational Fluid Dynamics., (M.D. Gunzburger & R.A. Nicolaides, eds), Cambridge University Press, 1993. pp. 17–65.

Fractional Step, le Θ scheme — II

- R. Rannacher
  Applications of mathematics in industry and technology, B.G. Teubner, Stuttgart, Germany, 1989. Chap. Numerical analysis of nonstationary fluid flow (a survey); pp. 34–53; (V.C. Boffi & H. Neunzert, eds).
- R. Rannacher
  Navier–Stokes equations: theory and numerical methods, Springer–Verlag, Berlin, Germany, 1990. Chap. On the numerical analysis of the non–stationary Navier–Stokes equations; pp. 180–193; (J. Heywood et al., eds).

Fractional Step, le Θ scheme — III

- S. Turek
  A comparative study of some time–stepping techniques for the incompressible Navier–Stokes equations: From fully implicit nonlinear schemes to semi–implicit projection methods, Int. J. Numer. Meth. Fluids, 22(10):987–1012, 1996.
- A. Smith & D.J. Silvester
  Implicit algorithms and their linearization for the transient incompressible Navier–Stokes equations, IMA J. Numer. Anal. 17: 527–545, 1997.