Fluid-structure interaction problems in the cardiovascular system

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Cardiovascular system modelling ???

- Blood flows and arteries are much more complicated than what will be presented in this talk !

Normal coronary artery
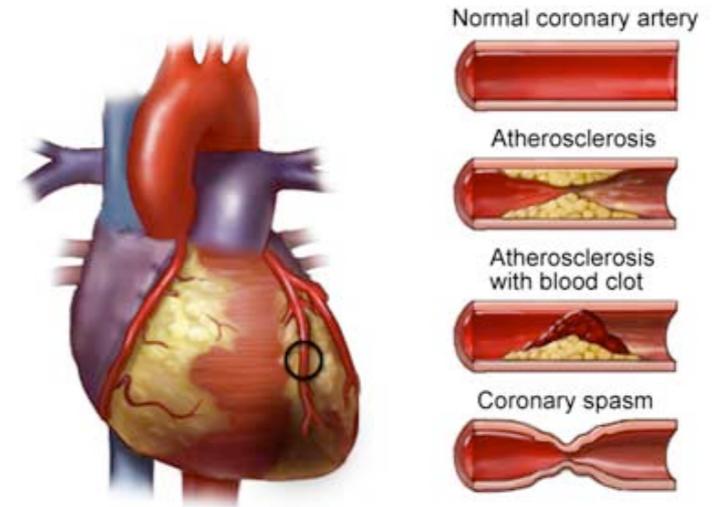
Atherosclerosis

Atherosclerosis with blood clot

Coronary spasm

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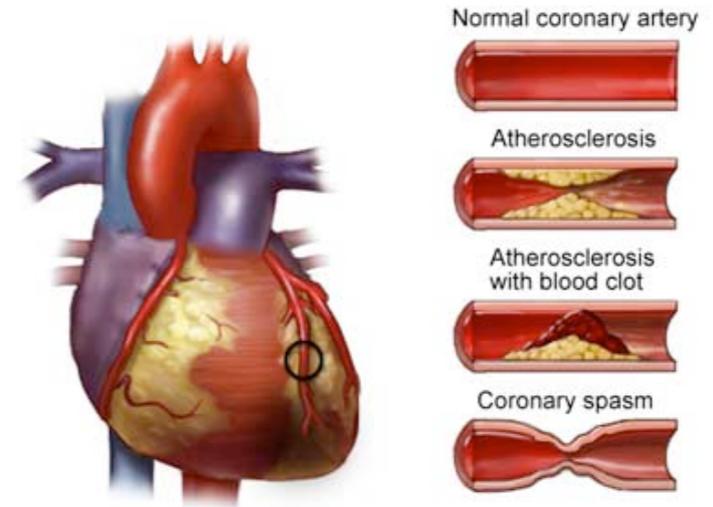
- Nevertheless : even with simplified models, mathematical modelling may help to improve some therapies or medical devices
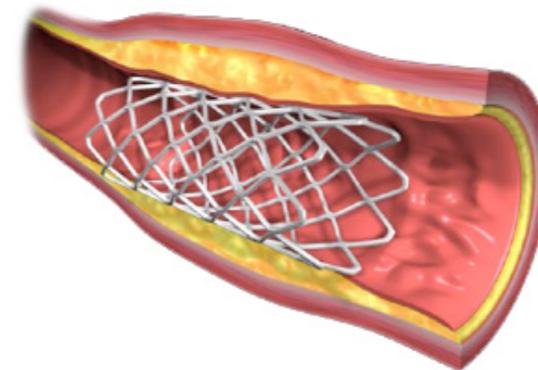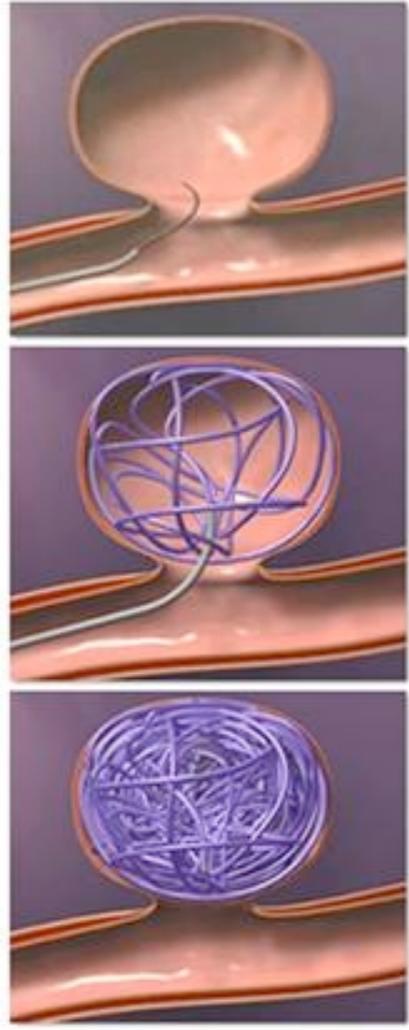
Ex 1: Stents and aneurisms
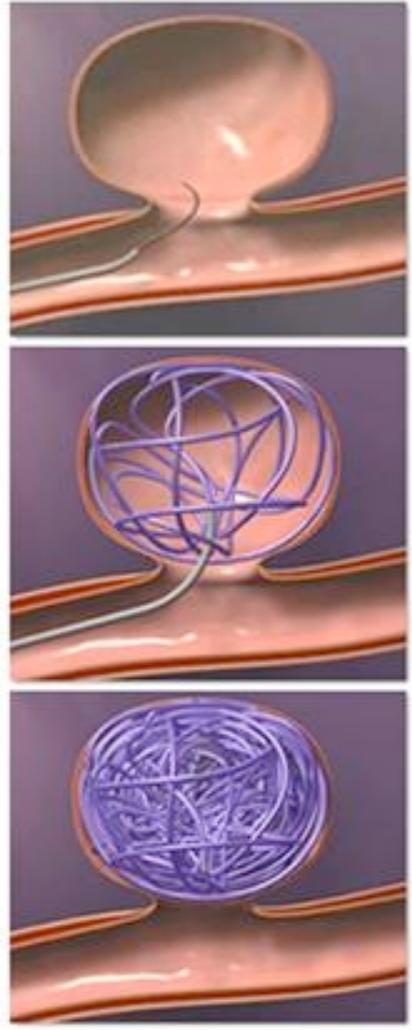
Coils

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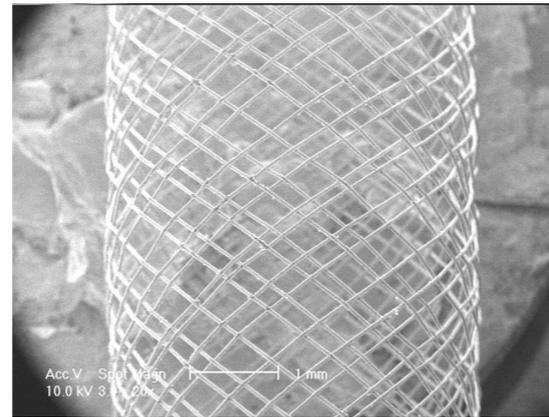
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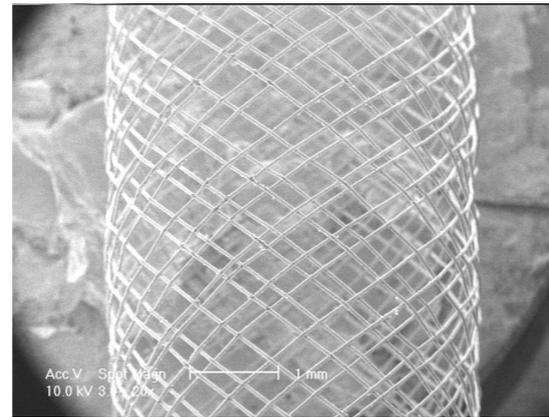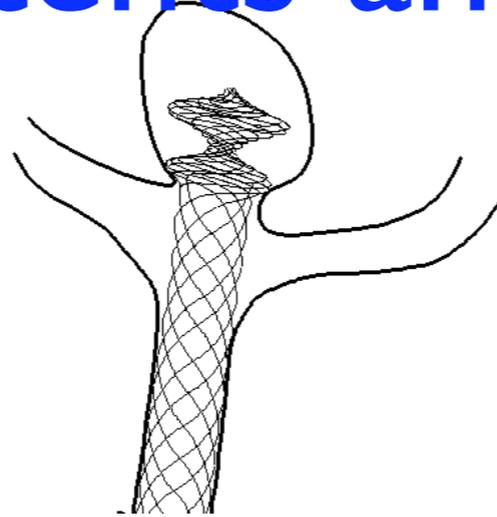
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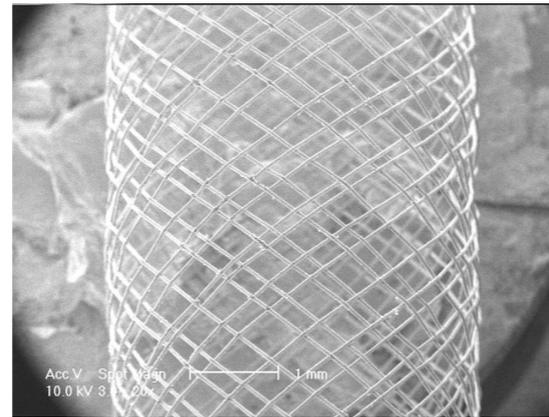
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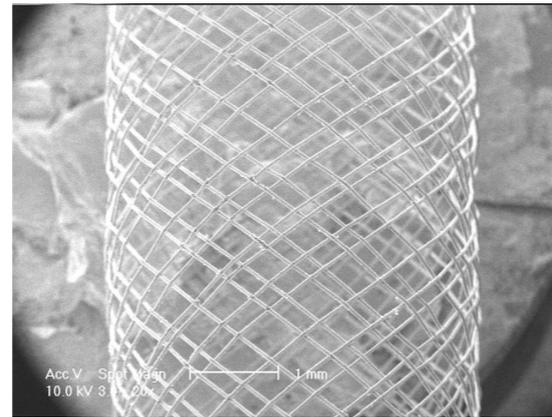
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In vivo experiments
(Cardiatis /INRA)

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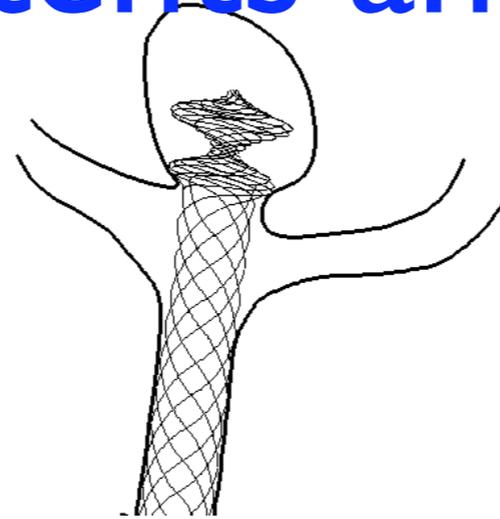
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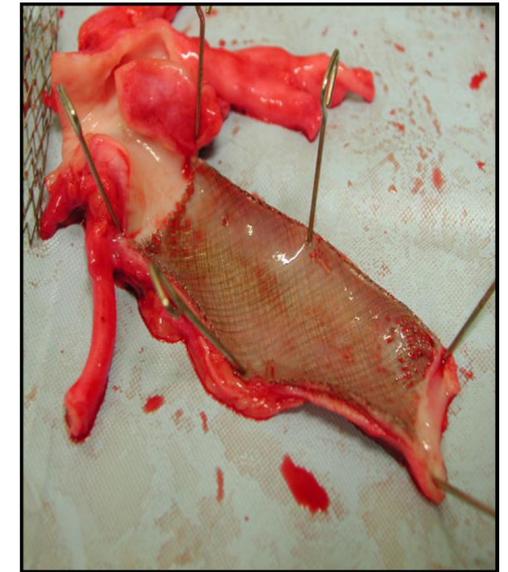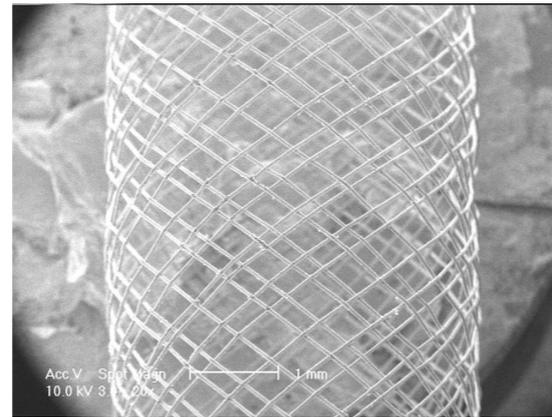
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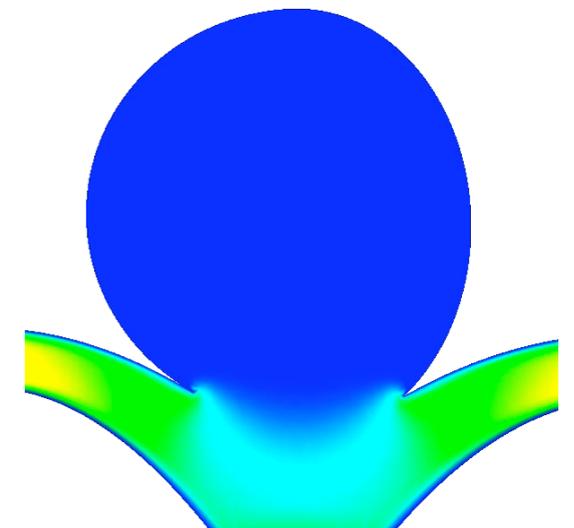
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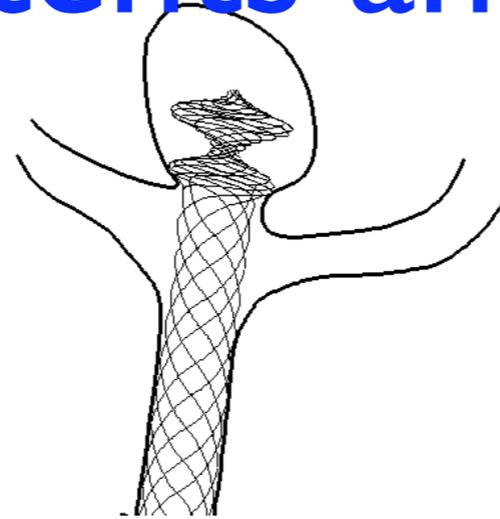
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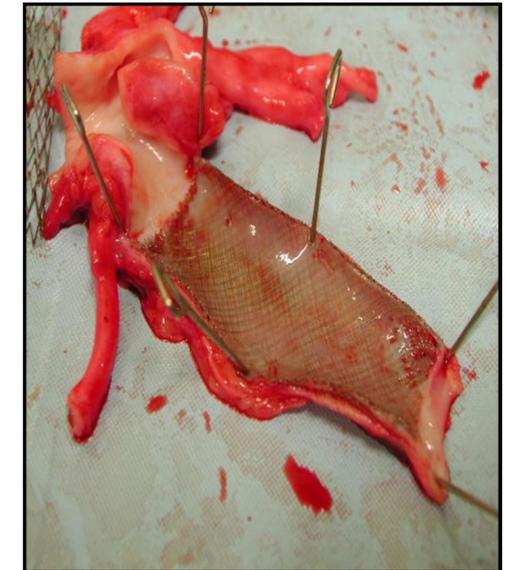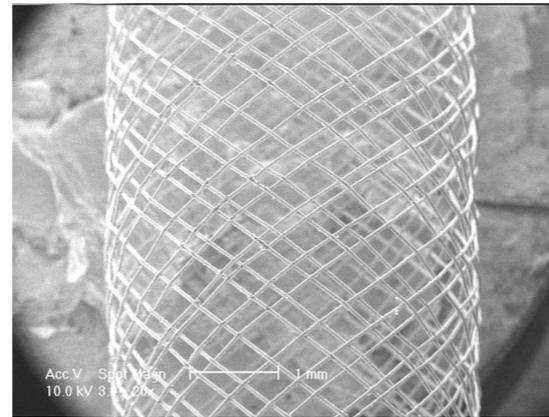
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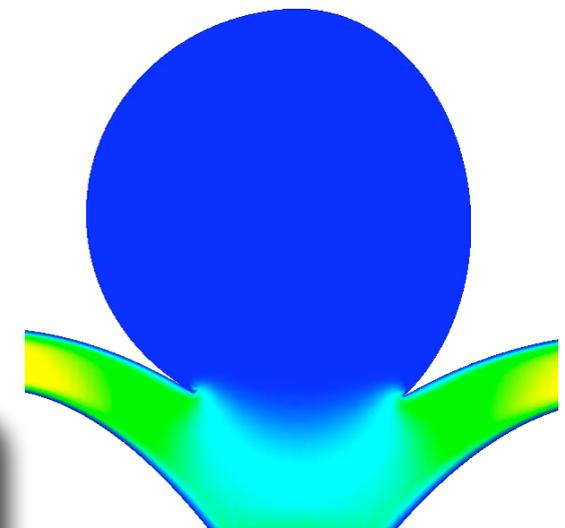
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- stent permeability/aneurysm blood flow?
- stent permeability/colateral arteries blood

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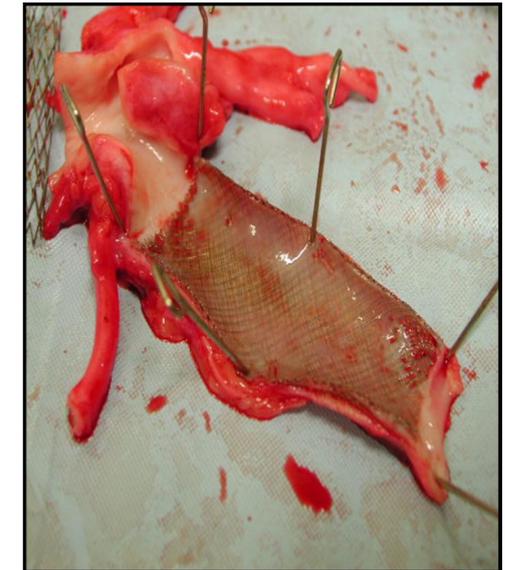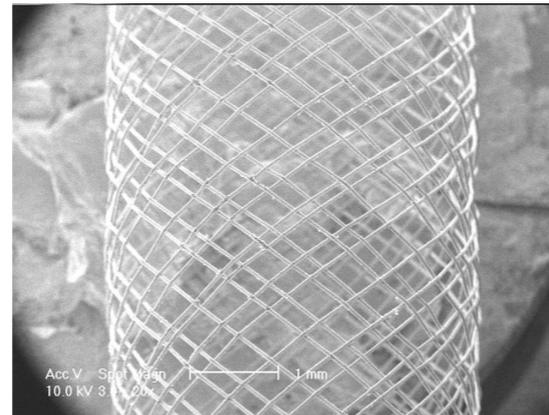
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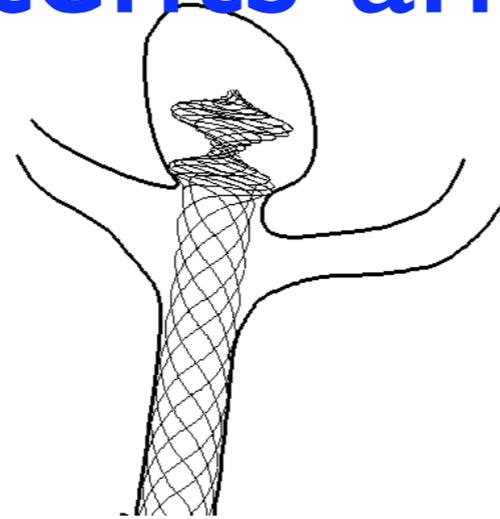
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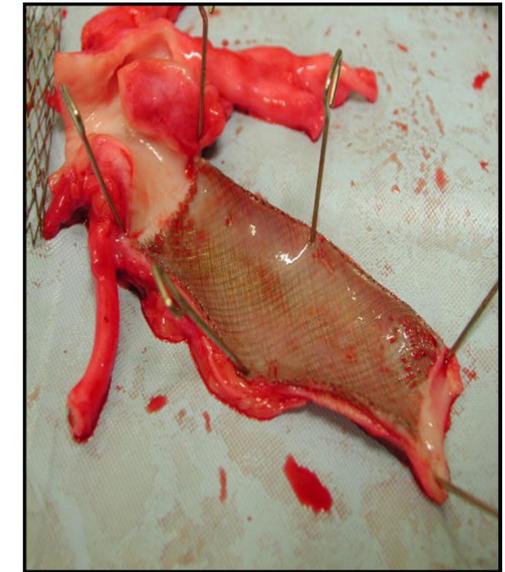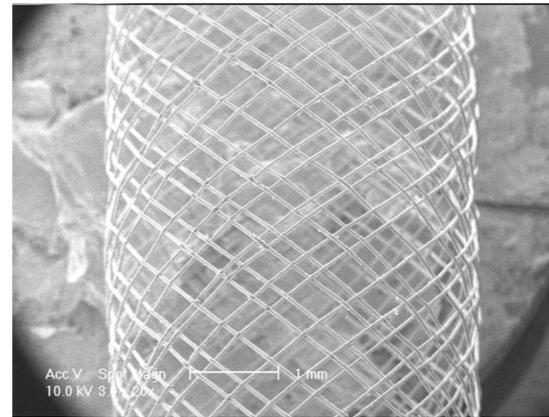
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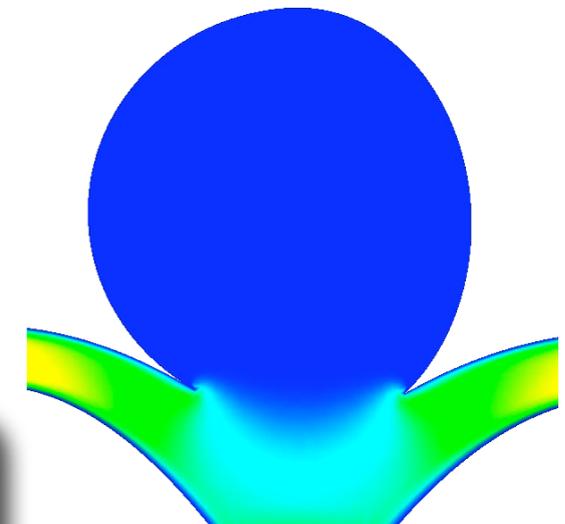
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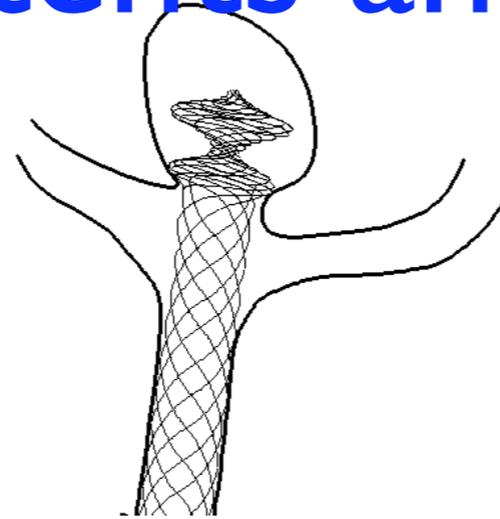
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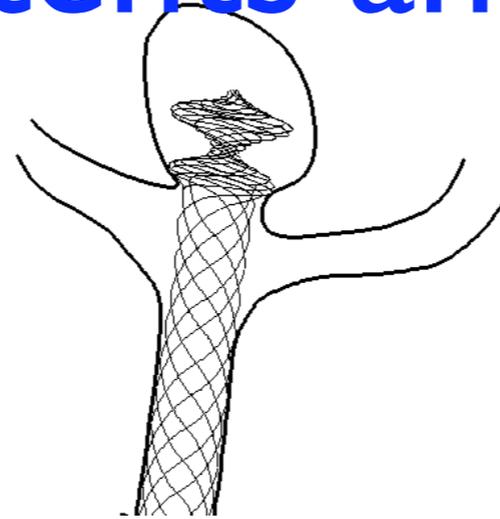
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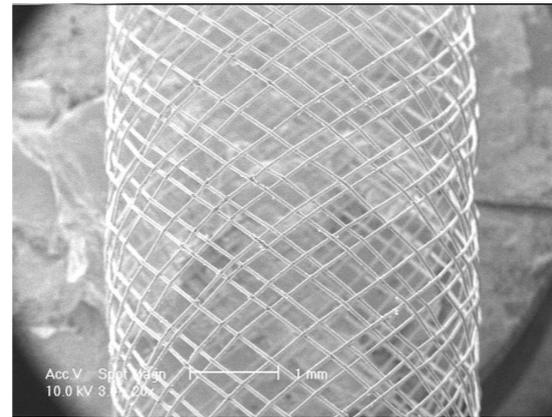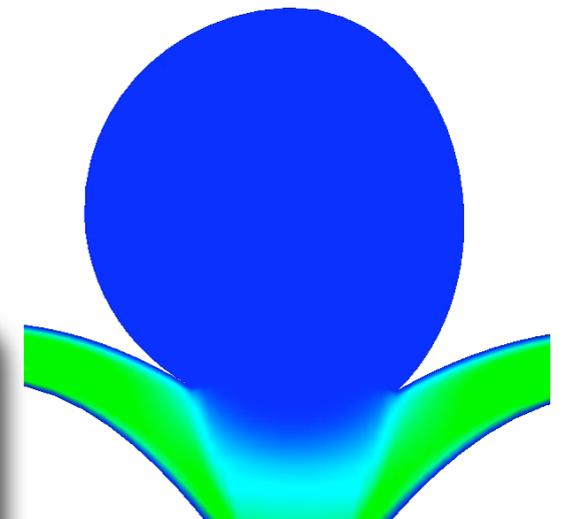
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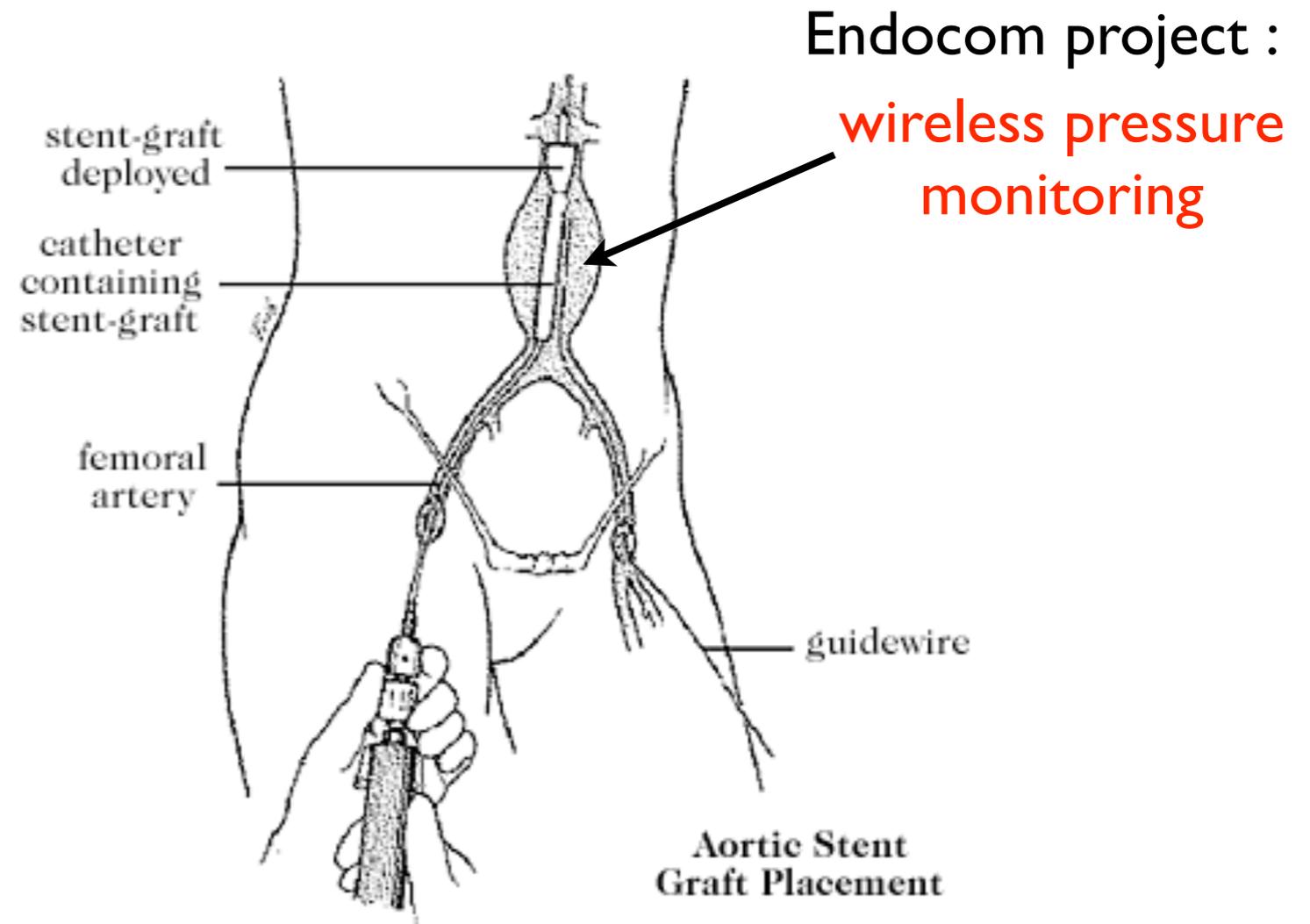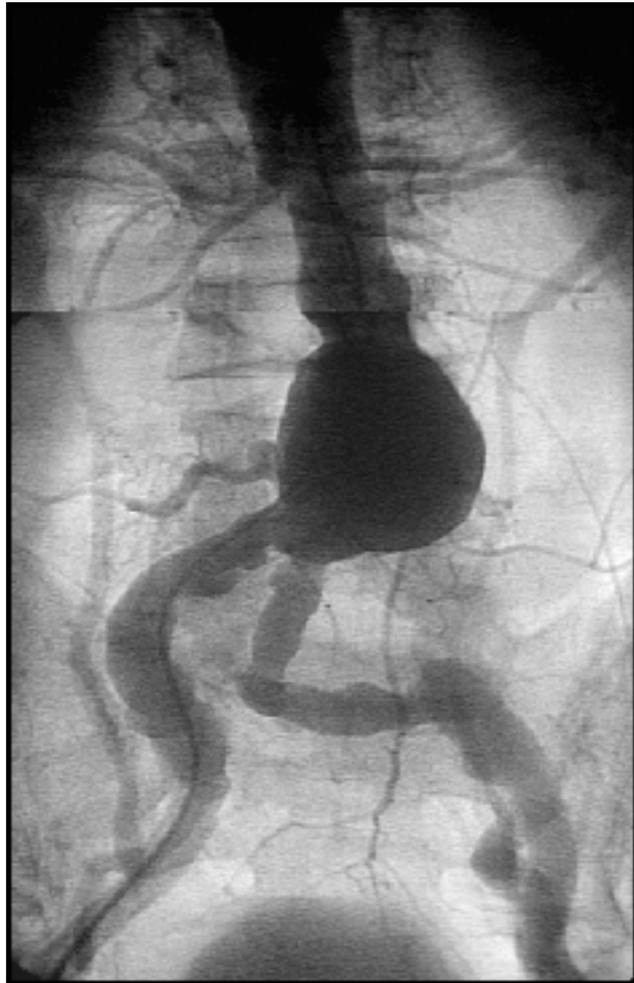
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« *In silico* » experiment
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With stent

Ex 2: Abdominal Aortic Aneurism

Endocom project :

wireless pressure monitoring

stent-graft deployed

catheter containing stent-graft

femoral artery

guidewire

Aortic Stent Graft Placement

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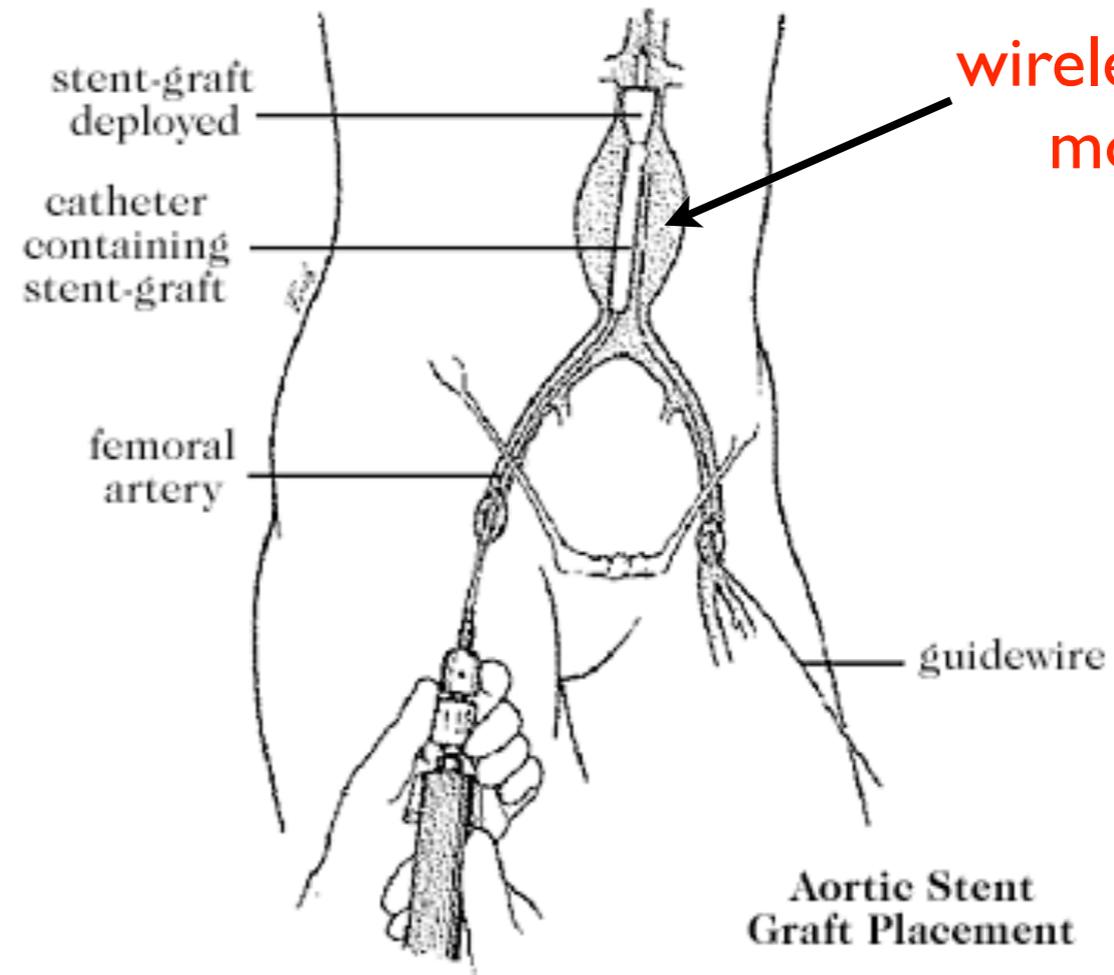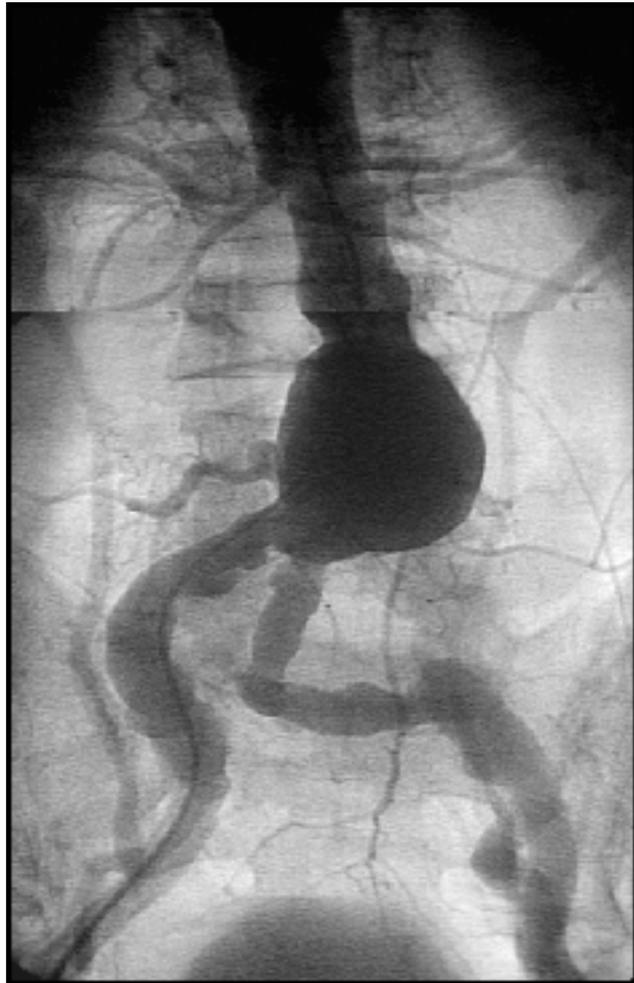
Numerical simulation:

- optimal position of the sensor

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  - Moving domain: geometrical non-linearities, topological change (contact)

Focus of this talk : fluid-structure interaction

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  - Moving domain: geometrical non-linearities, topological change (contact)

  - Boundary conditions

Outline

- Fluid-structure interaction in arteries

  - Stability issues - Strongly couple schemes

  - A stable weakly coupled scheme

- Fluid-structure interaction with valves

  - Fictitious domain method

  - FSI and Contact management

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Fluid-structure interaction in arteries

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Fluid-structure interaction in arteries

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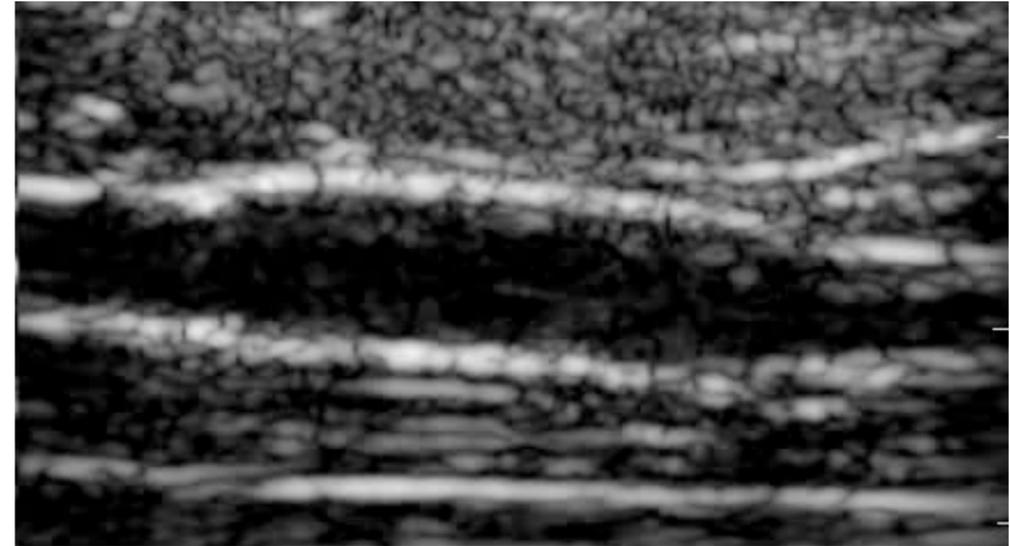
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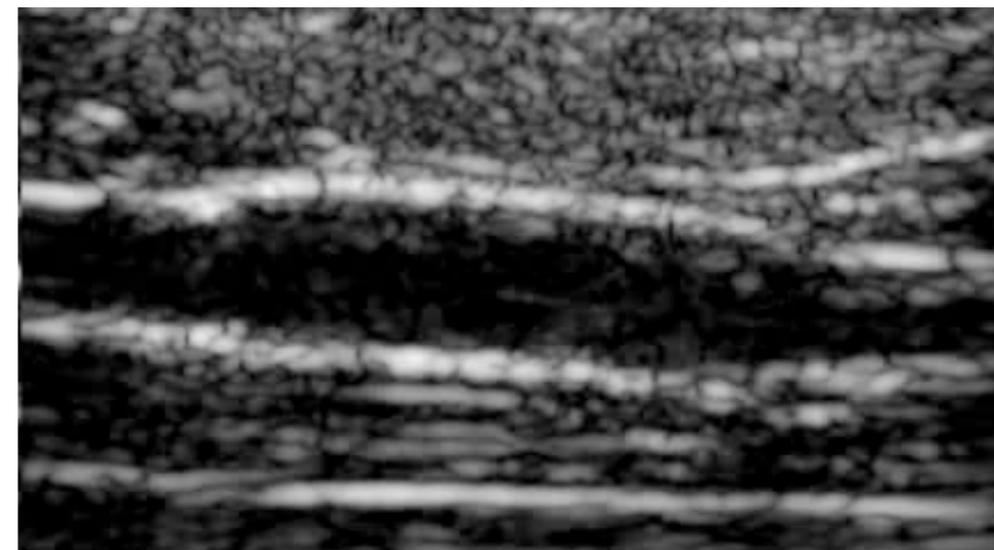
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Wall : Arbitrary Lagrangian Eulerian (ALE)

Fluid-structure interaction in arteries

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de Boer, van Zuijlen, Bijl, CMAME 2007
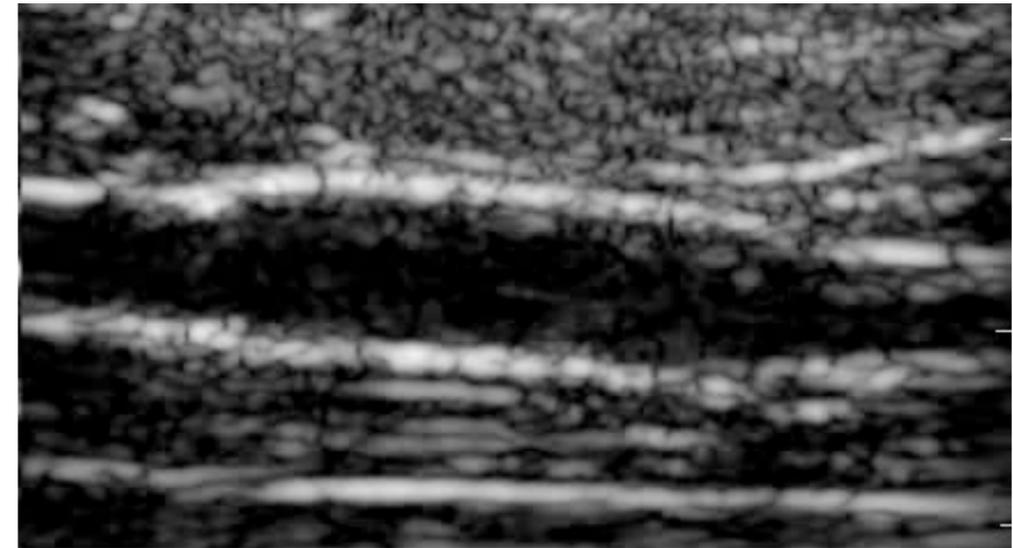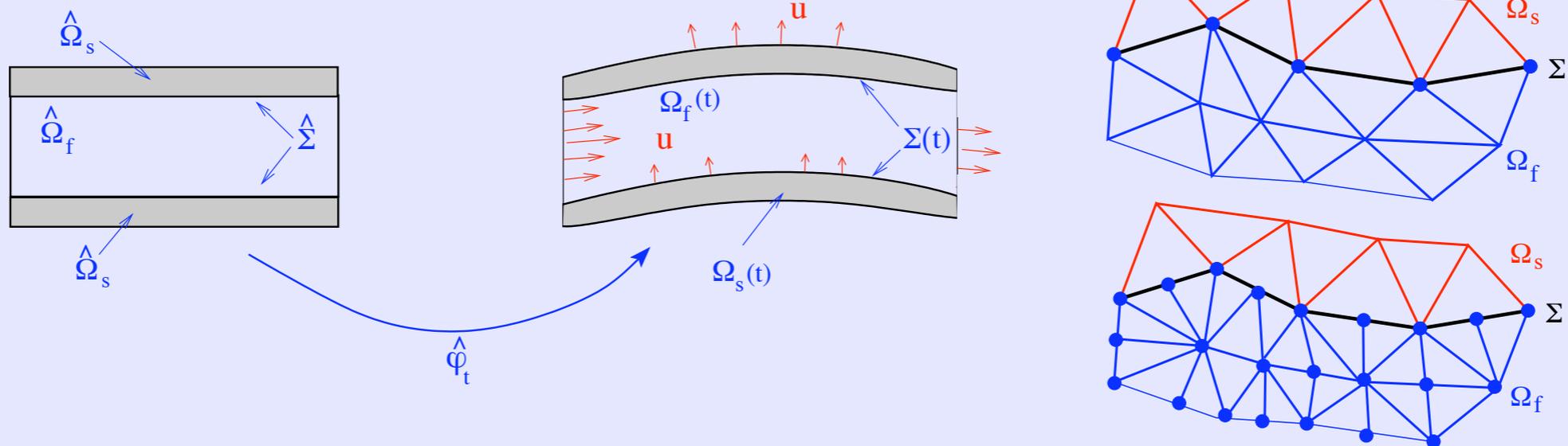
Geometrical description
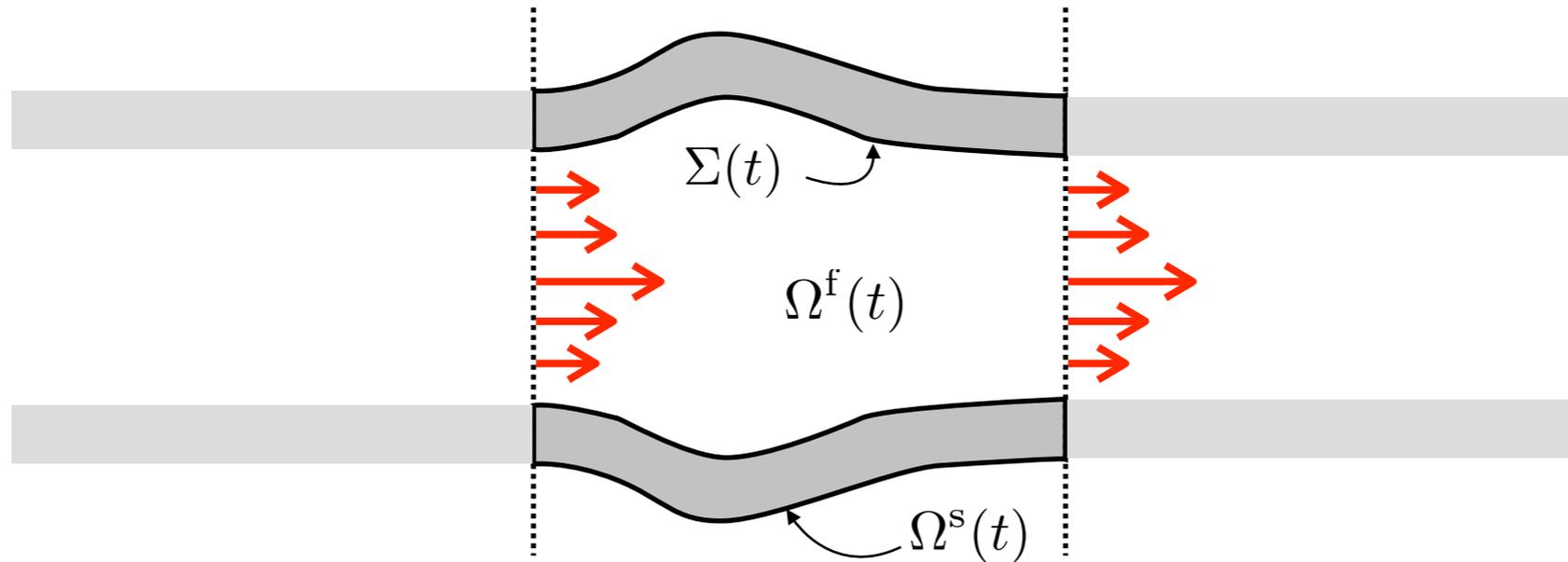
$\Omega^{\mathrm{f}}(t)$: moving fluid domain (filled with a viscous fluid : blood)

$\Omega^{\mathrm{s}}(t)$: solid current configuration (artery wall)

$\Sigma(t)$: fluid-solid interface, where we enforce

- continuity of velocity
- continuity of stress

The fluid-structure problem:

Determine $\Omega^{\mathrm{f}}(t)$, velocity and stress within the fluid and solid

The coupled problem

- Fluid equations:

$$\rho^{\mathrm{f}}\left(\frac{\partial \boldsymbol{u}}{\partial t}_{|\widehat{\boldsymbol{x}}} + (\boldsymbol{u} - \boldsymbol{w}) \cdot \boldsymbol{\nabla} \boldsymbol{u}\right) - 2\mu \mathbf{div}\boldsymbol{\epsilon}(\boldsymbol{u}) + \boldsymbol{\nabla} p = \boldsymbol{0}, \quad \text{in} \quad \Omega^{\mathrm{f}}(t)$$

$$\operatorname{div}\boldsymbol{u} = 0, \quad \text{in} \quad \Omega^{\mathrm{f}}(t)$$

$$\boldsymbol{\sigma}(\boldsymbol{u}, p)\boldsymbol{n} = \boldsymbol{g}, \quad \text{on} \quad \Gamma_{\mathrm{N}}^{\mathrm{f}}$$

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- Solid equations:

$$\rho^{\mathrm{s}}\frac{\partial^2 \boldsymbol{d}}{\partial t^2} - \mathbf{div}\big(\boldsymbol{F}(\boldsymbol{d})\boldsymbol{S}(\boldsymbol{d})\big) = \boldsymbol{0}, \quad \text{in} \quad \widehat{\Omega}^{\mathrm{s}}$$

$$\boldsymbol{d} = \boldsymbol{0}, \quad \text{on} \quad \widehat{\Gamma}^{\mathrm{s}}_{\mathrm{D}}$$

$$\boldsymbol{F}(\boldsymbol{d})\boldsymbol{S}(\boldsymbol{d})\widehat{\boldsymbol{n}}^{\mathrm{s}} = \boldsymbol{0}, \quad \text{on} \quad \widehat{\Gamma}^{\mathrm{s}}_{\mathrm{N}}$$

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- Coupling conditions:

$$\boldsymbol{d}^{\mathrm{f}} = \mathrm{Ext}(\boldsymbol{d}_{|\widehat{\Sigma}}), \quad \boldsymbol{w}(\boldsymbol{d}^{\mathrm{f}}) = \frac{\partial \boldsymbol{d}^{\mathrm{f}}}{\partial t} \quad \text{in} \quad \widehat{\Omega}^{\mathrm{f}}, \quad \Omega^{\mathrm{f}}(t) = (I + \boldsymbol{d}^{\mathrm{f}})(\widehat{\Omega}^{\mathrm{f}}), \quad \text{(geometry)}$$

$$\boldsymbol{u} = \boldsymbol{w}(\boldsymbol{d}^{\mathrm{f}}), \quad \text{on} \quad \Sigma(t), \quad \text{(velocity)}$$

$$\boldsymbol{F}(\boldsymbol{d})\boldsymbol{S}(\boldsymbol{d})\widehat{\boldsymbol{n}} = J(\boldsymbol{d}^{\mathrm{f}})\boldsymbol{\sigma}(\boldsymbol{u}, p)\boldsymbol{F}(\boldsymbol{d}^{\mathrm{f}})^{-\mathrm{T}}\widehat{\boldsymbol{n}}, \quad \text{on} \quad \widehat{\Sigma}, \quad \text{(stress)}$$

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  - for valves

Implementation issues

- Use independent solvers for fluid and structure :

  - Advantage: re-usability of state-of-the-art algorithms

  - Difficulties: possible troubles with the coupling algorithms

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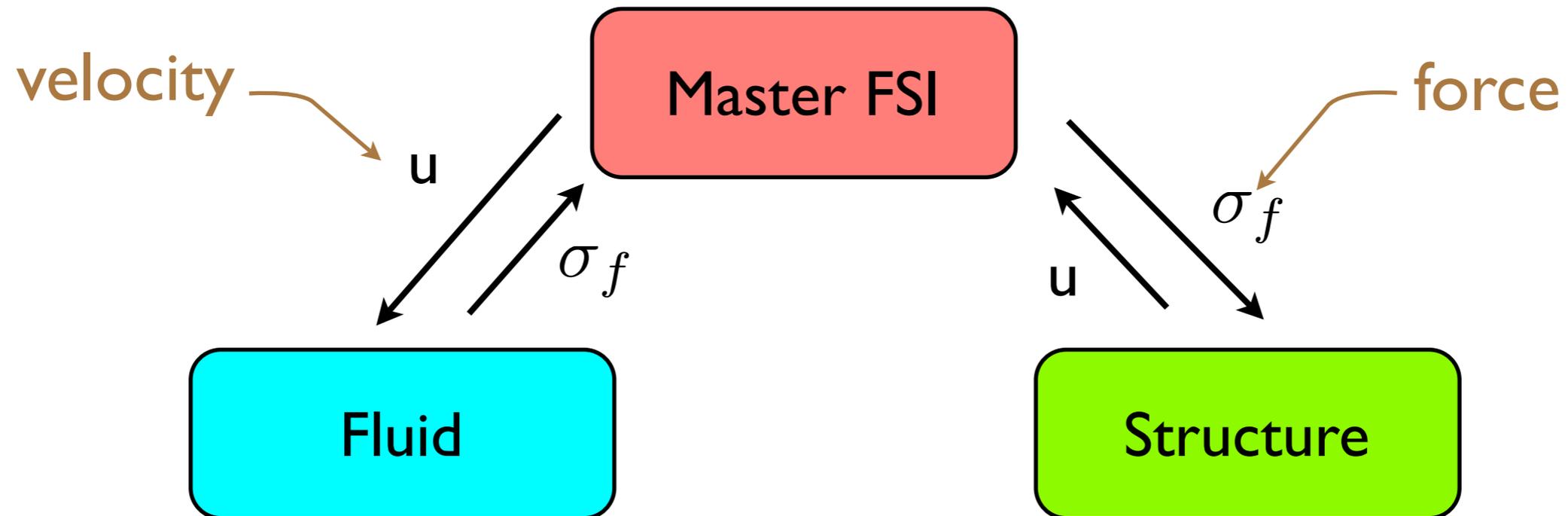
- **Strong** coupling: sub-iterations at each time step

- **Weak** coupling : 1 or 2 iterations per time step

Example: Dirichlet-Neumann

- Fluid sub-problem: $(\boldsymbol{d}^{\mathrm{f},n+1}, \boldsymbol{u}^{n+1}, p^{n+1}) = \mathcal{F}\big(\boldsymbol{d}^{n+1}_{|\widehat{\Sigma}}\big)$

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Fixed-point iterations with acceleration

(1) Initiallization:
$$\boldsymbol{\lambda}^0 = \boldsymbol{d}^n_{|\widehat{\Sigma}}$$

(2) Until convergence $(k \geq 0)$:

  (a) Solve fluid and solid:
$$\widetilde{\boldsymbol{\lambda}}^{k+1} = (\mathcal{S} \circ \mathcal{F})(\boldsymbol{\lambda}^k)$$

  (b) Relaxation:
$$\boldsymbol{\lambda}^{k+1} = \omega_k \widetilde{\boldsymbol{\lambda}}^{k+1} + (1 - \omega_k)\boldsymbol{\lambda}^k, \quad \omega_k \in (0, 1]$$

▸ **Aitken acceleration** *(Wall, Ramm, 2001)*:
$$\omega_k = \frac{(\boldsymbol{\lambda}^k - \boldsymbol{\lambda}^{k-1}) \cdot (\boldsymbol{\lambda}^k - \widetilde{\boldsymbol{\lambda}}^k - \boldsymbol{\lambda}^{k-1} + \widetilde{\boldsymbol{\lambda}}^{k-1})}{|\boldsymbol{\lambda}^k - \widetilde{\boldsymbol{\lambda}}^k - \boldsymbol{\lambda}^{k-1} + \widetilde{\boldsymbol{\lambda}}^{k-1}|^2}$$

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Vierendeels, Lanoye, Degroote, Verdonck, 2007
Vierendeels, 2006

Domain decomposition

Domain decomposition

Option 1 : decompose first then linearize

- Dirichlet-Neumann
  - Fixed-point : *Le Tallec-Mouro (1999) Wall-Ramm (2001), ...*
  - Newton : *Fernandez-Moubachir (2003), ...*
  - *Inexact* Newton: *Matthies-Steindorf (2003), JFG-Vidrascu (2003), Mischler-van Brummelen-de Borst (2005), Vierendeels (2006),...*
- Neumann - Neumann
  - *Deparis-Discacciati-Quarteroni (2005), ...*
- Robin - Neumann
  - *Badia-Nobile-Vergara (2007)*

Domain decomposition

Option 1 : decompose first then linearize

- Dirichlet-Neumann
  - Fixed-point : *Le Tallec-Mouro (1999) Wall-Ramm (2001), ...*
  - Newton : *Fernandez-Moubachir (2003), ...*
  - *Inexact* Newton: *Matthies-Steindorf (2003), JFG-Vidrascu (2003), Mischler-van Brummelen-de Borst (2005), Vierendeels (2006),...*
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Option 2 : linearize first then decompose

- *Fernandez, JFG, Gloria, Vidrascu (2007)*

Common feature:
*These algorithms are strongly coupled, for **stability**.*
Number of sub-iterations : between 10 and 100 !

Explicit coupling: some observations

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- An explicit algorithm is *a priori* very efficient...

$$\text{FSI cost} \approx \text{FLUID cost} + \text{SOLID cost}$$

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- Energy estimate with an artificial interface power term:

$$\int_{\Sigma^{n+1}} \boldsymbol{\sigma}(\boldsymbol{u}^{n+1}, p^{n+1})\boldsymbol{n} \cdot \left(\boldsymbol{u}^{n+1} - \frac{\boldsymbol{d}^{n+1} - \boldsymbol{d}^n}{\delta t} \right)$$

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- Explicit coupling is stable and widely used in aeroelasticity !
  Ex : *Farhat, van der Zee, Geuzaine, 2006*

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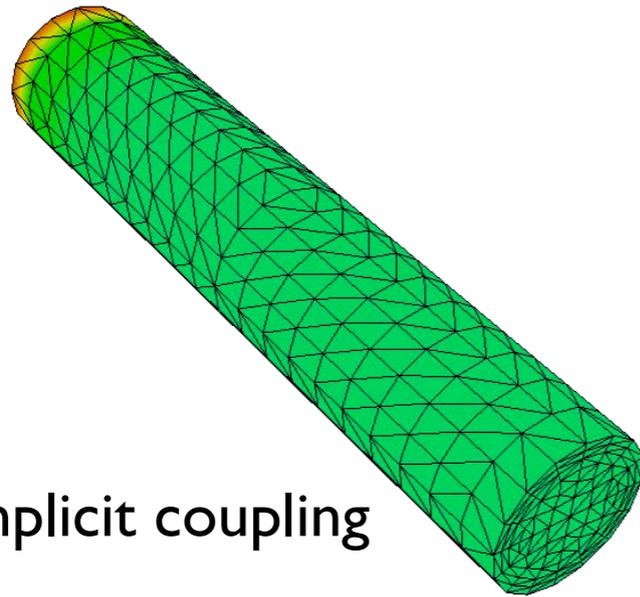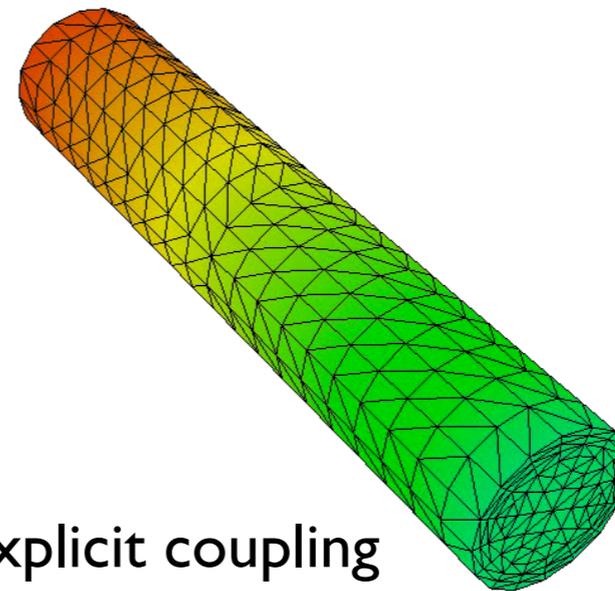
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▸ What is the **source of instabilities of those schemes in blood flows** ?

A 2D simplified model

(Causin, JFG, Nobile, 2004)
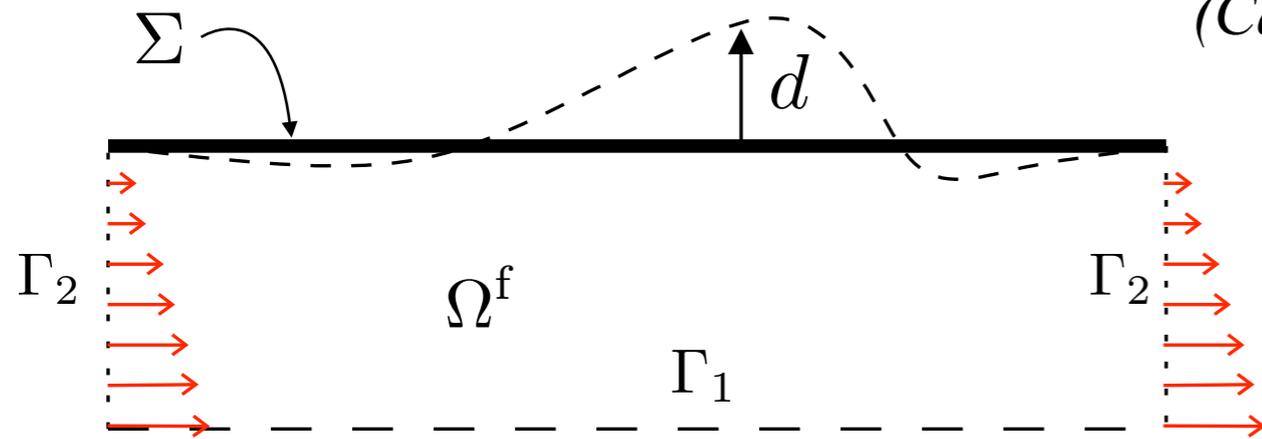
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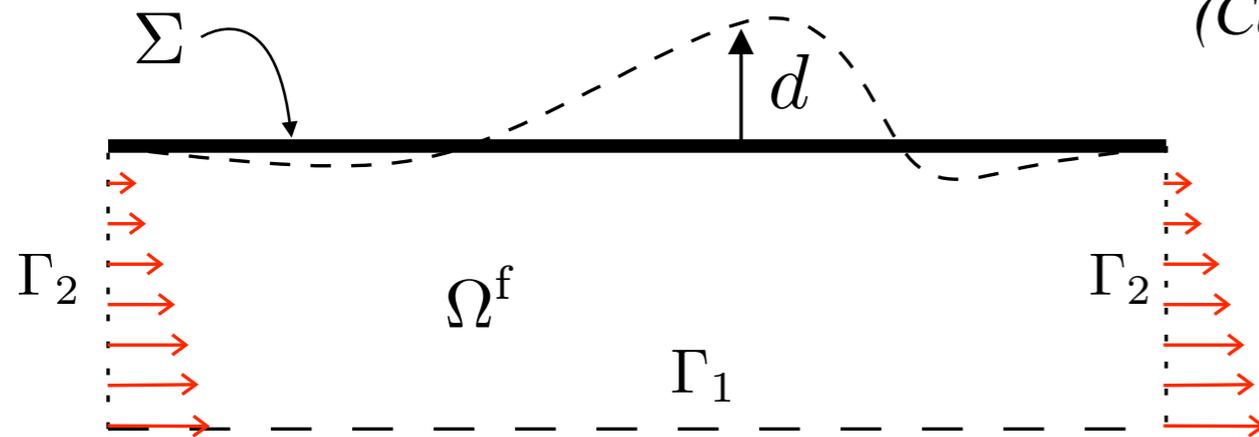
- Solid: string model (small displacements)

$$\rho^{\mathrm{s}}\varepsilon\ddot{d} + Ld = p_{|\Sigma}, \quad \text{in} \quad \Sigma,$$

with

- d: vertical displacement
- ε: vessel thickness
- L: linear operator (for instance $L\eta = a\eta - b\frac{\partial^2 \eta}{\partial x^2}$)

A 2D simplified model

(Causin, JFG, Nobile, 2004)

- Solid: string model (infinitesimal displacements)

$$\rho^{\text{s}} \varepsilon \ddot{d} + L d = p_{|\Sigma}, \quad \text{in} \quad \Sigma,$$

- Fluid: fixed fluid domain, no viscous/convective terms

$$
\begin{cases}
\rho^{\text{f}} \dfrac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\nabla} p = 0, & \text{in} \quad \Omega^{\text{f}} \\[2mm]
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Interest of this model :

▸ **Physics**: reproduces propagation phenomena

▸ **Numerics**: explicit coupling unstable

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(Causin, JFG, Nobile, 2004)

Σ

d

Γ_2 Ω^{f} Γ_2

Γ_1

implicit coupling

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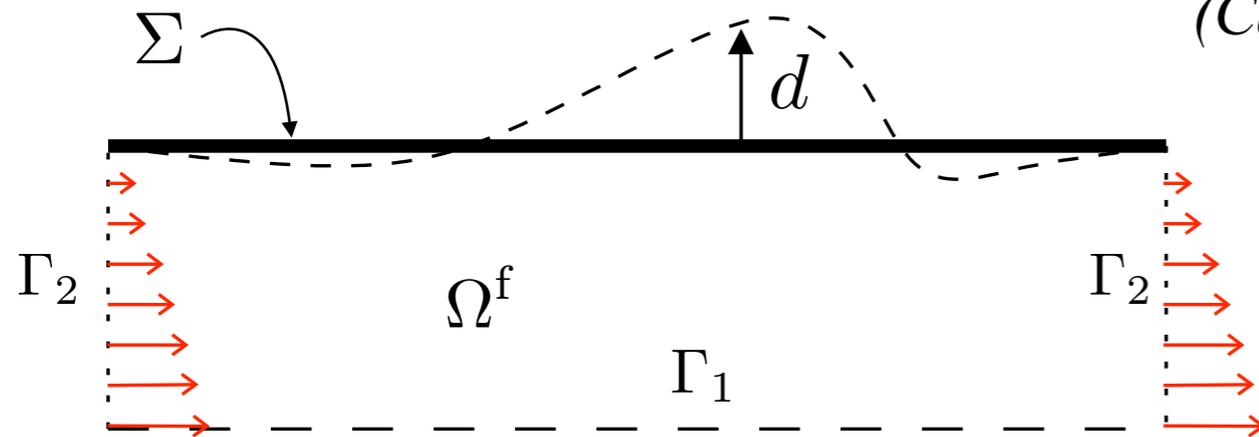
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The added-mass operator

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Theorem (Steklov-Poincaré operator)

The operator $\mathcal{M}_{\mathrm{A}} : H^{-\frac{1}{2}}(\Sigma) \to H^{\frac{1}{2}}(\Sigma)$ defined as: for each $g \in H^{-\frac{1}{2}}(\Sigma)$ we set $\mathcal{M}_{\mathrm{A}}(g) \stackrel{\text{def}}{=} q_{|\Gamma^{\mathrm{w}}}$, where $q \in H^1(\Omega^{\mathrm{f}})$ solves

$$\begin{cases} -\Delta q = 0, & \text{in} \quad \Omega^{\mathrm{f}} \\ \dfrac{\partial q}{\partial \boldsymbol{n}} = g, & \text{on} \quad \Sigma \\ \dfrac{\partial q}{\partial \boldsymbol{n}} = 0, & \text{on} \quad \Gamma_1 \\ q = 0, & \text{on} \quad \Gamma_2 \end{cases}$$

is a linear, compact, positive and self-adjoint operator in $L^2(\Sigma)$.

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✓ From this definition, we have $\quad p_{|\Sigma} = \mathcal{M}_{\mathrm{A}}(-\rho^{\mathrm{f}} \ddot{d}) = -\rho^{\mathrm{f}} \mathcal{M}_{\mathrm{A}} \ddot{d}$

The added-mass effect

Fluid: $\begin{cases} -\Delta p = 0, & \text{in} \quad \Omega^{\mathrm{f}} \\[2mm] \dfrac{\partial p}{\partial \boldsymbol{n}} = -\rho^{\mathrm{f}} \ddot{d}, & \text{on} \quad \Sigma \\[2mm] \dfrac{\partial p}{\partial \boldsymbol{n}} = 0, & \text{on} \quad \Gamma_1 \\[2mm] p = 0, & \text{on} \quad \Gamma_2 \end{cases}$

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$$p_{|\Sigma} = -\rho^{\mathrm{f}} \mathcal{M}_{\mathrm{A}} \ddot{d}$$

$$\left(\rho^{\mathrm{s}} \varepsilon + \rho^{\mathrm{f}} \mathcal{M}_{\mathrm{A}} \right) \ddot{d} + L d = 0, \quad \text{in} \quad \Sigma$$

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Remarks:

▸ This equation looks like a structure equation, except for the extra "mass" term

▸ The **fluid-structure coupling** can be condensed into an **extra mass** action on the structure (hence the terminology "added-mass effect")

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Solid: $\quad \rho^{\text{s}} \varepsilon \ddot{d} + Ld = p_{|\Sigma}, \quad \text{in} \quad \Sigma,$ (1)

$$p_{|\Sigma} = -\rho^{\text{f}} \mathcal{M}_{\text{A}} \ddot{d}$$

$$\left(\rho^{\text{s}} \varepsilon + \rho^{\text{f}} \mathcal{M}_{\text{A}} \right) \ddot{d} + Ld = 0, \quad \text{in} \quad \Sigma$$ (2)

Remarks:

▶ This equation looks like a structure equation, except for the extra "mass" term
▶ The **fluid-structure coupling** can be condensed into an **extra mass** action on the structure (hence the terminology "added-mass effect")

Question :

What kind of time integration scheme of (2) arises from the explicit coupling of (1)?

Explicit coupling and added-mass

Fluid:
$$\begin{cases} \rho^{\mathrm{f}} \dfrac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^n}{\delta t} + \nabla p^{n+1} = 0 \\[2ex] \operatorname{div} \boldsymbol{u}^{n+1} = 0 \\[2ex] \boldsymbol{u}^{n+1} \cdot \boldsymbol{n} = \dfrac{{\color{red}d^n} - d^{n-1}}{\delta t} \end{cases}$$

Explicit coupling and added-mass

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Solid:
$$\rho^{\mathrm{s}} \varepsilon \frac{d^{n+1} - 2d^n + d^{n-1}}{\delta t^2} + L d^{n+1} = p^{n+1}_{|\Sigma}$$

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$$
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Condensed FSI problem:

$$\rho^{\mathrm{s}} \varepsilon \frac{d^{n+1} - 2d^n + d^{n-1}}{\delta t^2} + \rho^{\mathrm{f}} \mathcal{M}_{\mathrm{A}} \frac{d^n - 2d^{n-1} + d^{n-2}}{\delta t^2} + L \eta^{n+1} = 0$$

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$$

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$$

Condensed FSI problem:

$$
\rho^{\mathrm{s}} \varepsilon \underbrace{\frac{d^{n+1} - 2d^n + d^{n-1}}{\delta t^2}}_{\textbf{implicit}} + \rho^{\mathrm{f}} \mathcal{M}_{\mathrm{A}} \frac{d^n - 2d^{n-1} + d^{n-2}}{\delta t^2} + L \eta^{n+1} = 0
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Explicit coupling and added-mass

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Condensed FSI problem:
$$
\rho^{\mathrm{s}} \varepsilon \underbrace{\frac{d^{n+1} - 2d^n + d^{n-1}}{\delta t^2}}_{\textbf{implicit}} + \rho^{\mathrm{f}} \mathcal{M}_{\mathrm{A}} \underbrace{\frac{d^n - 2d^{n-1} + d^{n-2}}{\delta t^2}}_{\textbf{explicit}} + L \eta^{n+1} = 0
$$

Explicit coupling and added-mass

Fluid:
$$\begin{cases} \rho^{\mathrm{f}} \dfrac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^n}{\delta t} + \nabla p^{n+1} = 0 \\[2mm] \operatorname{div} \boldsymbol{u}^{n+1} = 0 \\[2mm] \boldsymbol{u}^{n+1} \cdot \boldsymbol{n} = \dfrac{d^n - d^{n-1}}{\delta t} \end{cases} \xRightarrow{\operatorname{div}} \begin{cases} -\Delta p^{n+1} = 0 \\[2mm] \dfrac{\partial p^{n+1}}{\partial \boldsymbol{n}} = -\rho^{\mathrm{f}} \dfrac{d^n - 2d^{n-1} + d^{n-2}}{\delta t^2} \end{cases}$$

Solid:
$$\rho^{\mathrm{s}} \varepsilon \frac{d^{n+1} - 2d^n + d^{n-1}}{\delta t^2} + L d^{n+1} = p^{n+1}_{|\Sigma} \qquad \boxed{p^{n+1}_{|\Sigma} = -\rho^{\mathrm{f}} \mathcal{M}_{\mathrm{A}} \frac{d^n - 2d^{n-1} + d^{n-2}}{\delta t^2}}$$

Condensed FSI problem:

$$\underbrace{\rho^{\mathrm{s}} \varepsilon \frac{d^{n+1} - 2d^n + d^{n-1}}{\delta t^2}}_{\textbf{implicit}} + \underbrace{\rho^{\mathrm{f}} \mathcal{M}_{\mathrm{A}} \frac{d^n - 2d^{n-1} + d^{n-2}}{\delta t^2}}_{\textbf{explicit}} + L \eta^{n+1} = 0$$

Observation:

Weak coupling leads to an explicit discretization of the added-mass

An unconditional instability result

Proposition *(Causin-JFG-Nobile 04)*

Let λ_{max} be the largest eigenvalue of $\mathcal{M}_{\mathrm{A}}$ and assume that $L\eta = a\eta$. Then, the previous explicit coupling scheme is unconditionally unstable whenever

$$\frac{\rho^{\mathrm{f}} \lambda_{\mathrm{max}}}{\rho^{\mathrm{s}} \varepsilon} \geq 1. \tag{1}$$

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$$\frac{\rho^{\mathrm{f}}\lambda_{\mathrm{max}}}{\rho^{\mathrm{s}}\varepsilon} \geq 1. \tag{1}$$

Remarks:

▸ The instability condition does not depend on the time step

▸ The instability condition confirms two numerical observations:
  • Instabilities might occur when the structure is **light**, **thin** and **slender**
  • In aeroelasticity $\rho^{\mathrm{f}} \ll \rho^{\mathrm{s}}$, hence weak (*i.e.* explicit) coupling is stable

Implicit / Explicit coupling: summary

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- Implicit coupling stable but too expensive

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Three ideas: *(Fernandez, JFG, Grandmont, 2006)*
- ▶ Treat implicitly the added-mass effect (incompressibility, pressure stress)
- ▶ Treat explicitly the fluid domain motion, convective and viscous effects
- ▶ Perform this using a projection scheme (Chorin-Teman) within the fluid

The Chorin-Teman projection scheme

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Main feature:
Incompressibility and viscous/convective effects are decoupled

The Chorin-Teman projection scheme

- Incompressible Navier-Stokes equations:

$$\rho^{\mathrm{f}} \left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla}\boldsymbol{u} \right) - 2\mu \mathbf{div}\boldsymbol{\epsilon}(\boldsymbol{u}) + \boldsymbol{\nabla}p = \boldsymbol{0}, \quad \mathrm{in} \quad \Omega^{\mathrm{f}}$$

$$\mathrm{div}\,\boldsymbol{u} = 0, \quad \mathrm{in} \quad \Omega^{\mathrm{f}}$$

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- **Viscous** step:

$$\left\{
\begin{aligned}
\rho^{\mathrm{f}}\left(\frac{\widetilde{\boldsymbol{u}}^{n+1} - \boldsymbol{u}^n}{\delta t} + \widetilde{\boldsymbol{u}}^{n+1}\cdot\boldsymbol{\nabla}\widetilde{\boldsymbol{u}}^{n+1}\right) - 2\mu\,\mathrm{div}\,\boldsymbol{\epsilon}(\widetilde{\boldsymbol{u}}^{n+1}) = 0, \quad \text{in} \quad \Omega \\
\widetilde{\boldsymbol{u}}^{n+1} = 0, \quad \text{on} \quad \partial\Omega
\end{aligned}
\right.$$

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\left\{
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\end{aligned}
\right.
$$

- **Projection** step:

$$
\left\{
\begin{aligned}
&\rho^{\mathrm{f}}\frac{\boldsymbol{u}^{n+1} - \widetilde{\boldsymbol{u}}^{n+1}}{\delta t} + \boldsymbol{\nabla} p^{n+1} = 0, \quad \text{in} \quad \Omega \\
&\operatorname{div}\boldsymbol{u}^{n+1} = 0, \quad \text{in} \quad \Omega \\
&\boldsymbol{u}^{n+1} \cdot \boldsymbol{n} = 0, \quad \text{on} \quad \partial\Omega
\end{aligned}
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Semi-implicit coupling: explicit part

- Viscous sub-step:

$$\boldsymbol{d}^{\mathrm{f},n+1} = \mathrm{Ext}(\boldsymbol{d}^n_{|\widehat{\Sigma}}), \quad \boldsymbol{w}^{n+1} = \frac{\boldsymbol{d}^{\mathrm{f},n+1} - \boldsymbol{d}^n}{\delta t}, \quad \Omega^{\mathrm{f},n+1} = (I + \boldsymbol{d}^{\mathrm{f},n+1})(\widehat{\Omega}^{\mathrm{f}}),$$

$$\rho^{\mathrm{f}} \left(\frac{\widetilde{\boldsymbol{u}}^{n+1} - \boldsymbol{u}^n}{\delta t} + (\widetilde{\boldsymbol{u}}^{n+1} - \boldsymbol{w}^{n+1}) \cdot \boldsymbol{\nabla} \widetilde{\boldsymbol{u}}^{n+1} \right) - 2\mu \mathbf{div}\, \boldsymbol{\epsilon}(\widetilde{\boldsymbol{u}}^{n+1}) = 0, \quad \text{in} \quad \Omega^{\mathrm{f},n+1}$$

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$$\rho^{\mathrm{f}} \left(\frac{\widetilde{u}^{n+1} - u^n}{\delta t} + (\widetilde{u}^{n+1} - w^{n+1}) \cdot \boldsymbol{\nabla} \widetilde{u}^{n+1} \right) - 2\mu \, \mathbf{div}\, \boldsymbol{\epsilon}(\widetilde{u}^{n+1}) = 0, \quad \text{in} \quad \Omega^{\mathrm{f},n+1}$$

$$\widetilde{u}^{n+1} = w^{n+1}, \quad \text{on} \quad \Sigma^{n+1}$$

Observation:

▸ Fluid domain, viscous and convective effects explicitly treated

Semi-implicit coupling: implicit part

- Fluid projection sub-step (in a known domain):

$$
\begin{cases}
\rho^{\mathrm{f}} \dfrac{\boldsymbol{u}^{n+1} - \widetilde{\boldsymbol{u}}^{n+1}}{\delta t} + \boldsymbol{\nabla} p^{n+1} = 0, & \text{in} \quad \Omega^{\mathrm{f},n+1} \\[2mm]
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\end{cases}
$$

- Solid equation:

$$
\begin{cases}
\rho^{\mathrm{s}} \dfrac{\boldsymbol{d}^{n+1} - 2\boldsymbol{d}^n + \boldsymbol{d}^{n-1}}{\delta t^2} - \mathbf{div}\left(\boldsymbol{F}(\boldsymbol{d}^{n+1})\boldsymbol{S}(\boldsymbol{d}^{n+1})\right) = \boldsymbol{0}, & \text{in} \quad \widehat{\Omega}^{\mathrm{s}} \\[2mm]
\boldsymbol{F}(\boldsymbol{d}^{n+1})\boldsymbol{S}(\boldsymbol{d}^{n+1})\widehat{\boldsymbol{n}} = J(\boldsymbol{d}^{\mathrm{f},n+1})\boldsymbol{\sigma}(\widetilde{\boldsymbol{u}}^{n+1}, p^{n+1})\boldsymbol{F}(\boldsymbol{d}^{\mathrm{f},n+1})^{-\mathrm{T}}\widehat{\boldsymbol{n}}, & \text{on} \quad \widehat{\Sigma}
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\end{cases}
\quad \overset{\operatorname{div}}{\Longrightarrow} \quad
\begin{cases}
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\end{cases}
$$

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Observations:

Semi-implicit coupling: implicit part

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Observations:

▸ Projection sub-step in a fixed fluid domain (fixed matrix)

Semi-implicit coupling: implicit part

- Fluid projection sub-step (in a known domain):

$$\begin{cases} \rho^{\mathrm{f}} \dfrac{\boldsymbol{u}^{n+1} - \widetilde{\boldsymbol{u}}^{n+1}}{\delta t} + \boldsymbol{\nabla} p^{n+1} = 0, & \text{in} \quad \Omega^{\mathrm{f},n+1} \\[2mm] \operatorname{div} \boldsymbol{u}^{n+1} = 0, & \text{in} \quad \Omega^{\mathrm{f},n+1} \\[2mm] \boldsymbol{u}^{n+1} \cdot \boldsymbol{n} = \dfrac{\boldsymbol{d}^{n+1} - \boldsymbol{d}^n}{\delta t} \cdot \boldsymbol{n}, & \text{on} \quad \Sigma^{n+1} \end{cases} \xrightarrow{\operatorname{div}} \begin{cases} -\Delta p^{n+1} = -\dfrac{\rho^{\mathrm{f}}}{\delta t} \operatorname{div} \widetilde{\boldsymbol{u}}^{n+1}, & \text{in} \quad \Omega^{\mathrm{f},n+1} \\[2mm] \dfrac{\partial p^{n+1}}{\partial \boldsymbol{n}} = -\rho^{\mathrm{f}} \dfrac{\boldsymbol{d}^{n+1} - 2\boldsymbol{d}^n + \boldsymbol{d}^{n-1}}{\delta t^2}, & \text{on} \quad \Sigma^{n+1} \end{cases}$$

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Observations:

‣ Projection sub-step in a fixed fluid domain (fixed matrix)

‣ Implicit part solved with cheaper (inner) iterations

A stability result (linear case)

Theorem: *(Fernandez-JFG-Grandmont 2006)*

Assume the interface matching operator to be L^2-stable. Then, under condition

$$\rho^{\mathrm{s}} \geq C \left(\rho^{\mathrm{f}} \frac{h}{H^{\alpha}} + 2\frac{\mu \delta t}{h H^{\alpha}} \right), \quad \text{with} \quad \alpha \overset{\mathrm{def}}{=} \begin{cases} 0, & \text{if} \quad \overline{\Omega^{\mathrm{s}}} = \Sigma, \\ 1, & \text{if} \quad \overline{\Omega^{\mathrm{s}}} \neq \Sigma, \end{cases}$$

the following discrete energy inequality holds:

$$\frac{1}{\delta t} \left[\frac{\rho^{\mathrm{f}}}{2} \|\boldsymbol{u}_h^{n+1}\|_{0,\Omega^{\mathrm{f}}}^2 - \frac{\rho^{\mathrm{f}}}{2} \|\boldsymbol{u}_h^{n}\|_{0,\Omega^{\mathrm{f}}}^2 + \frac{\rho^{\mathrm{s}}}{2} \left\| \frac{\boldsymbol{d}_H^{n+1} - \boldsymbol{d}_H^{n}}{\delta t} \right\|_{0,\Omega^{\mathrm{f}}}^2 - \frac{\rho^{\mathrm{s}}}{2} \left\| \frac{\boldsymbol{d}_H^{n} - \boldsymbol{d}_H^{n-1}}{\delta t} \right\|_{0,\Omega^{\mathrm{f}}}^2 \right]$$

$$+ \frac{1}{2\delta t} \left[a^{\mathrm{s}}(\boldsymbol{d}_H^{n+1}, \boldsymbol{d}_H^{n+1}) - a^{\mathrm{s}}(\boldsymbol{d}_H^{n}, \boldsymbol{d}_H^{n}) \right] + \mu \|\epsilon(\widetilde{\boldsymbol{u}}_h^{n+1})\|_{0,\Omega^{\mathrm{f}}}^2 \leq 0$$

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$$+ \frac{1}{2\delta t} \left[a^{\mathrm{s}}(\boldsymbol{d}_H^{n+1}, \boldsymbol{d}_H^{n+1}) - a^{\mathrm{s}}(\boldsymbol{d}_H^n, \boldsymbol{d}_H^n) \right] + \mu \|\epsilon(\widetilde{\boldsymbol{u}}_h^{n+1})\|_{0,\Omega^{\mathrm{f}}}^2 \leq 0$$

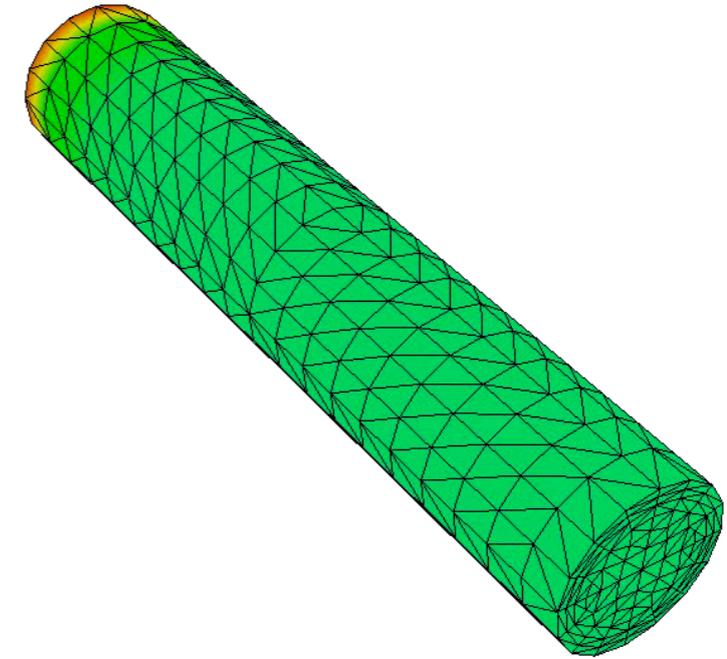Therefore, the semi-implicit coupling scheme is conditionnally stable in the energy norm.

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$$+ \frac{1}{2\delta t} \left[a^{\mathrm{s}}(\boldsymbol{d}_H^{n+1}, \boldsymbol{d}_H^{n+1}) - a^{\mathrm{s}}(\boldsymbol{d}_H^n, \boldsymbol{d}_H^n) \right] + \mu \|\epsilon(\widetilde{\boldsymbol{u}}_h^{n+1})\|_{0,\Omega^{\mathrm{f}}}^2 \leq 0$$

Therefore, the semi-implicit coupling scheme is conditionnally stable in the energy norm.

Remark: this semi-implicit algorithm has been extended by *Badia, Quarteroni, Quaini, 2007* to other projection schemes.

3D Navier-Sokes / Non-linear Shell coupling

- Straight cylinder: 50 time steps of length $\delta t = 2 \times 10^{-4} s$

COUPLING	ALGORITHM	CPU time	
Implicit	FP-Aitken	24.86	← 2001
	quasi-Newton	6.05	← 2003
	Newton	4.77	← 2003
Semi-Implicit	Newton	1	← 2006

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2001
2003
2006
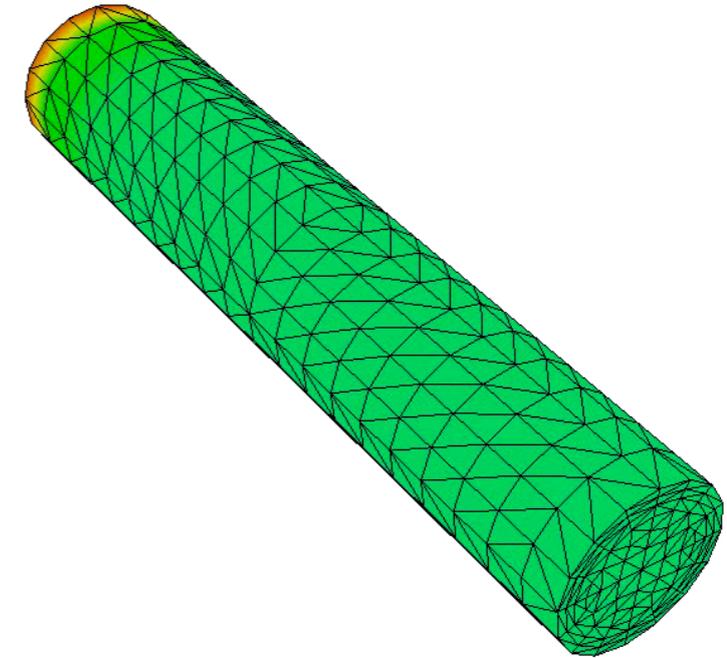
- Cerebral aneurysm: 20 time steps of length $\delta t = \times 10^{-3} s$

COUPLING	CPU time
Implicit	4.70
Semi-Implicit	1
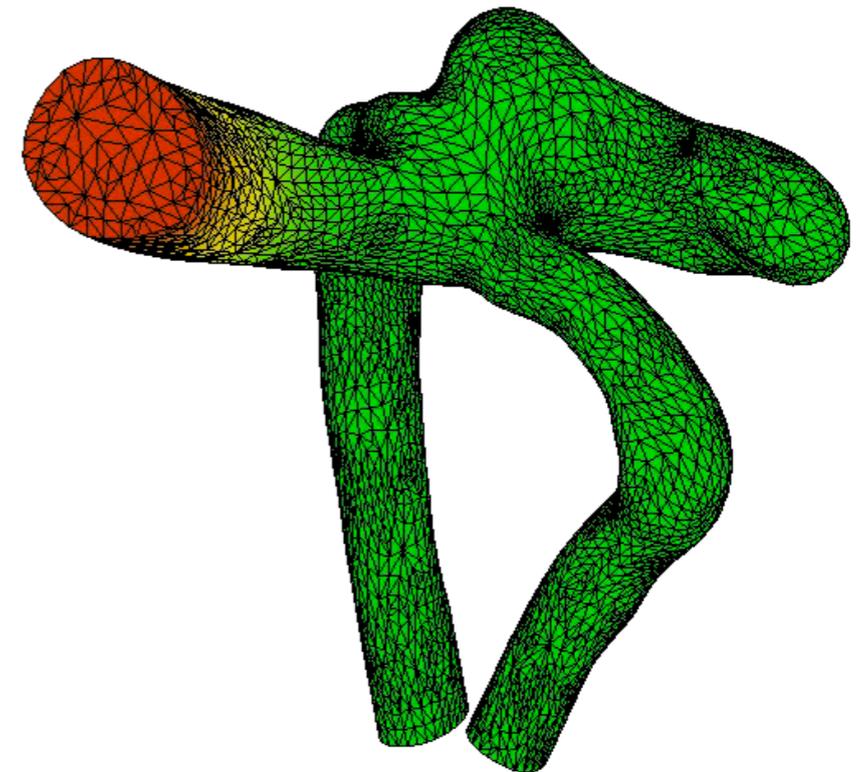
3D Navier-Sokes / Non-linear Shell coupling

- Abdominal aortic aneurysm (in-vitro model): 2 cardiac cycles, 1000 times steps
  - $\delta t = 1.68 \times 10^{-3} s$

  - Fluid: 26950 Hexahedra ($\mathbb{Q}_1/\mathbb{Q}_1$ FE)

  - Solid: 2240 Quadrilaterals (MITC4 FE)

  - Parameters: $\mu = 0.035\, poise$, $\rho^f = 1\, g/cm^3$, $\rho^s = 1.2\, g/cm^3$, $E = 610^6\, dynes/cm^2$, $\nu = 0.3$

Out-flow
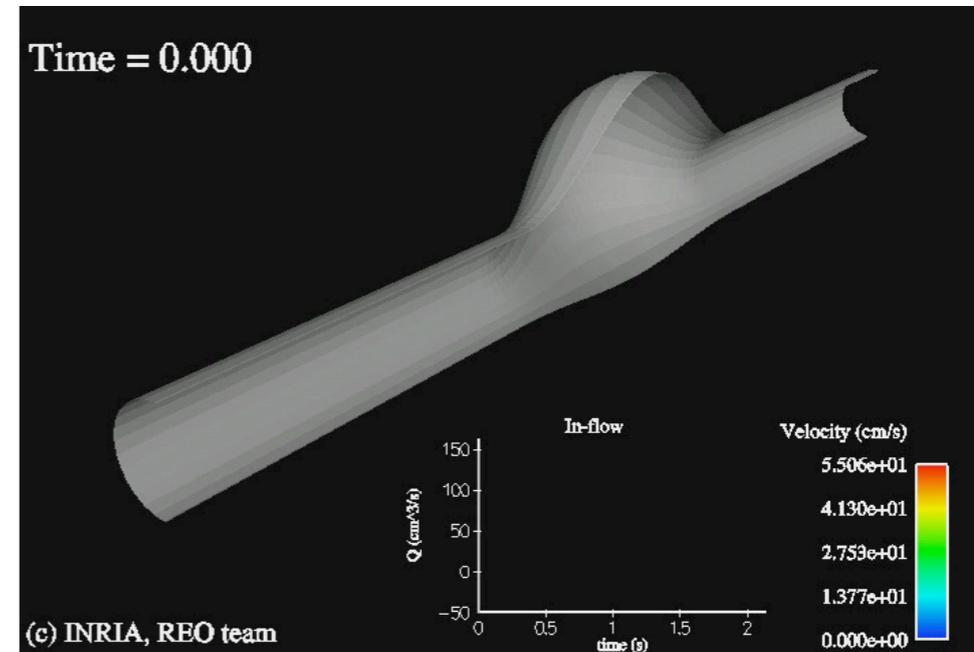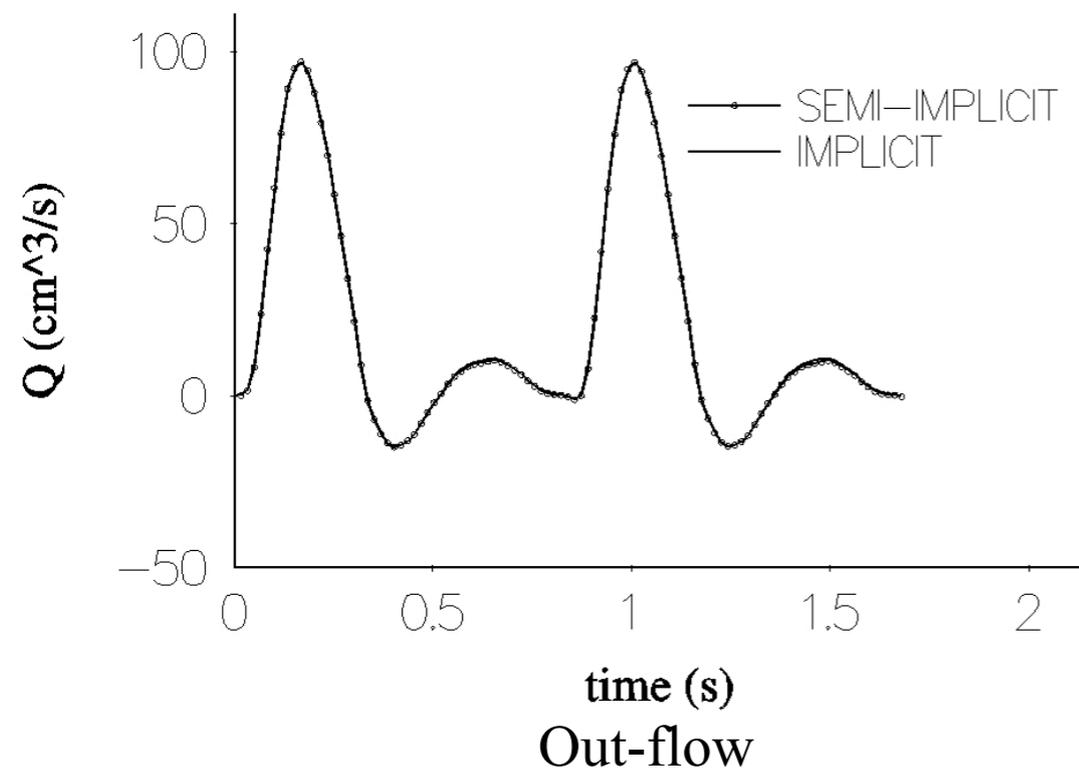
COUPLING	CPU time
Implicit	**9.3**
Semi-Implicit	**1.0**

Dimensionless CPU time

3D Navier-Sokes / Non-linear Shell coupling

- Carotid artery (in-vivo model): 9 cardiac cycles, 4500 times steps

  - $\delta t = 1.68 \times 10^{-3} s$

  - Fluid: 70047 Tetrahedra ($\mathbb{P}_1/\mathbb{P}_1$ FE)

  - Solid: 8103 Quadrilaterals (MITC4 FE)

  - Parameters: $\mu = 0.035\, poise$, $\rho^f = 1\, g/cm^3$, $\rho^s = 1.2\, g/cm^3$, $E = 6 \times 10^6\, dynes/cm^2$, $\nu = 0.3$.

Out-flow

COUPLING	CPU time
Implicit	**6.7**
Semi-Implicit	**1.0**

Dimensionless CPU time

3D Navier-Sokes / Non-linear Shell coupling

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Out-flow

COUPLING	CPU time
Implicit	**6.7**
Semi-Implicit	**1.0**

Dimensionless CPU time

Remarks on boundary conditions

- Spurious reflexion of pressure wave: 3D-1D coupling

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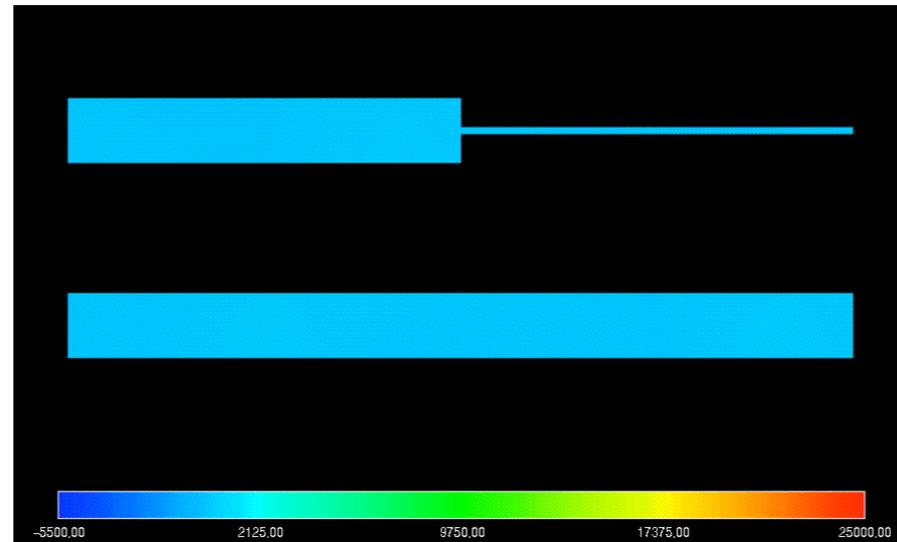
Remarks on boundary conditions

- Spurious reflexion of pressure wave: 3D-1D coupling
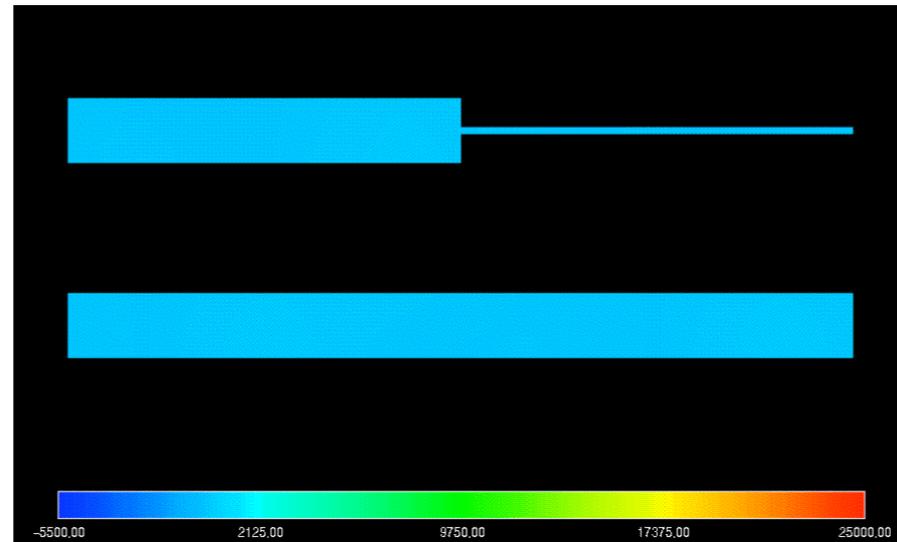
Formaggia, JFG, Quarteroni, Nobile, 2001
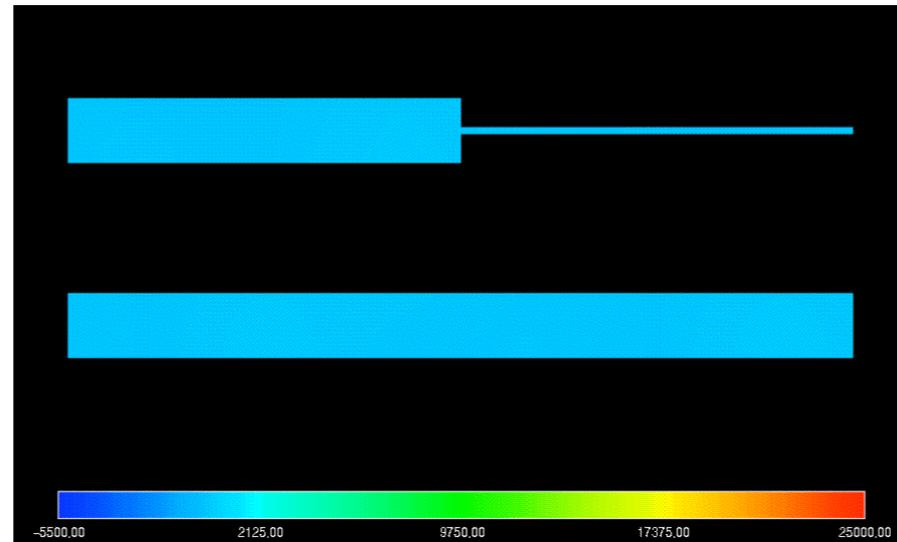*Formaggia, Moura, Nobile, 2007*
Formaggia, Veneziani, Vergara, 2006

Remarks on boundary conditions

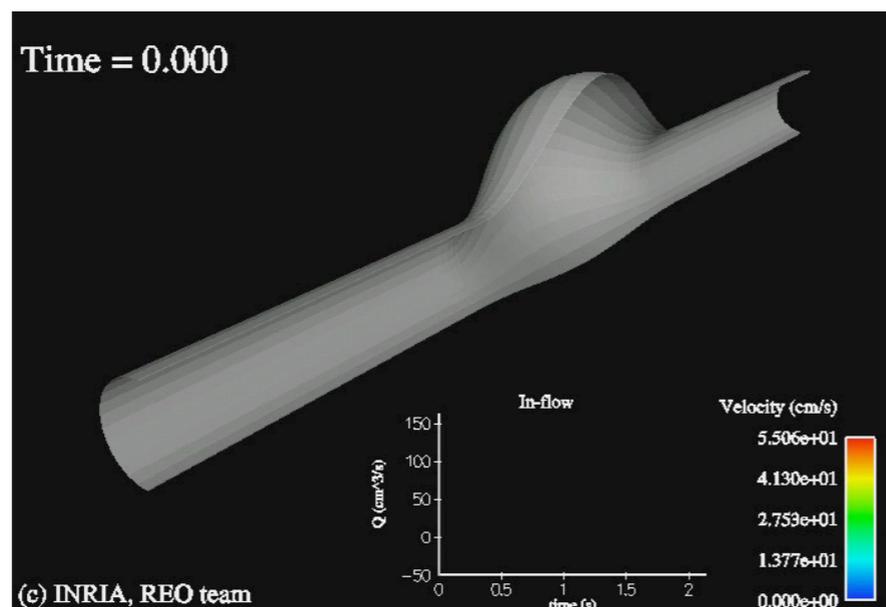- Spurious reflexion of pressure wave: 3D-1D coupling

Formaggia, JFG, Quarteroni, Nobile, 2001
Formaggia, Moura, Nobile, 2007
Formaggia, Veneziani, Vergara, 2006

- But pressure wave reflexion is not the only issue

Remarks on boundary conditions

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Formaggia, JFG, Quarteroni, Nobile, 2001
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Formaggia, JFG, Quarteroni, Nobile, 2001
Formaggia, Moura, Nobile, 2007
Formaggia, Veneziani, Vergara, 2006

- But pressure wave reflexion is not the only issue

The best non-reflecting outlet boundary condition cannot prevent the global (non physiological) bending !

Remarks on boundary conditions

Remarks on boundary conditions

- Surrounding tissues play a key role

Remarks on boundary conditions

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- Typical b.c. on the external part of the vessel :

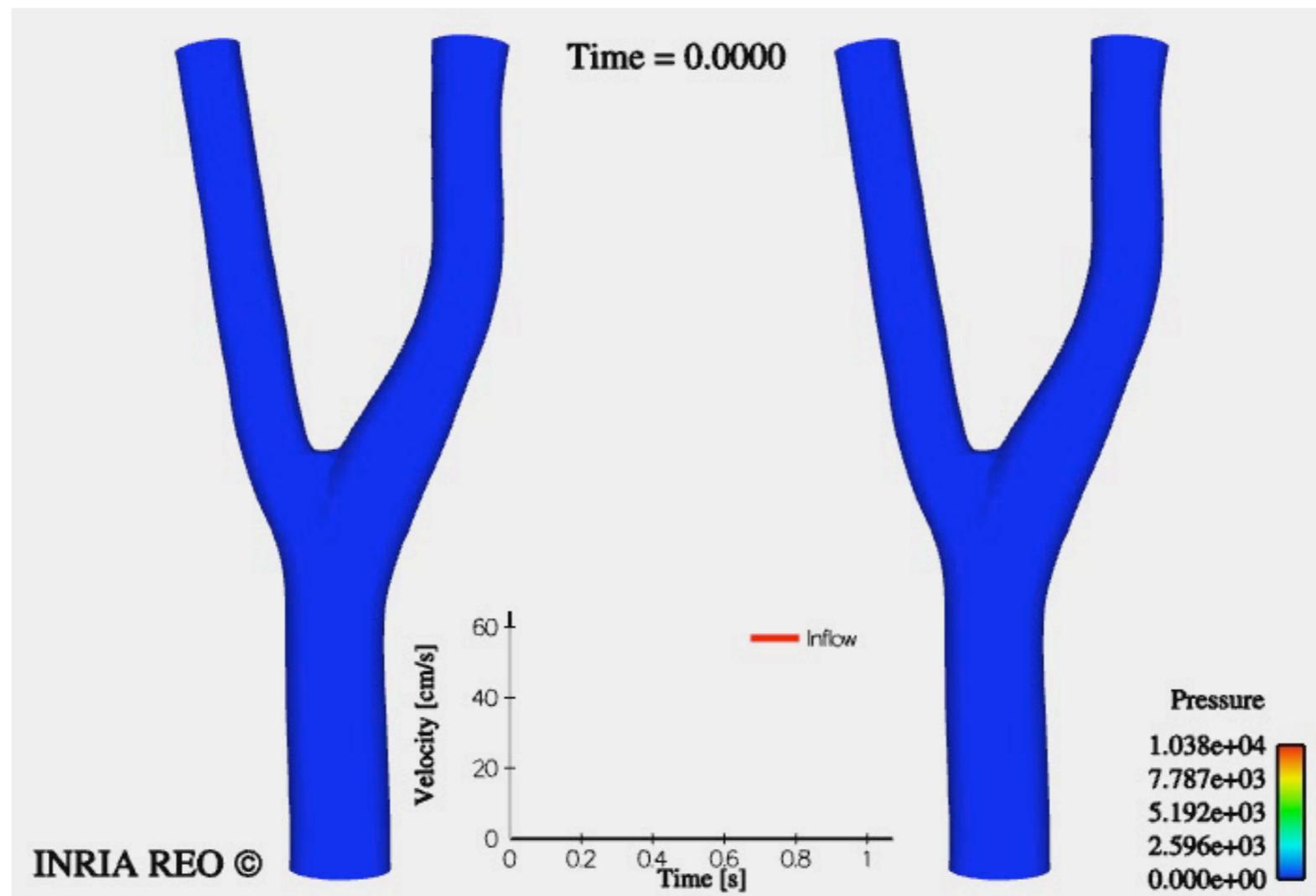$$\boldsymbol{F}(\boldsymbol{d})\ \boldsymbol{S}(\boldsymbol{d})\ \widehat{\boldsymbol{n}} = p_0$$

Remarks on boundary conditions

- Surrounding tissues play a key role

- Typical b.c. on the external part of the vessel :

$$\boldsymbol{F}(\boldsymbol{d})\ \boldsymbol{S}(\boldsymbol{d})\ \widehat{\boldsymbol{n}} = p_0$$

- A simple and affordable way to model the surrounding tissues ("silent blocks"):

$$\boldsymbol{F}(\boldsymbol{d})\ \boldsymbol{S}(\boldsymbol{d})\ \widehat{\boldsymbol{n}} = -k_s\boldsymbol{d} - c_s\frac{\partial \boldsymbol{d}}{\partial t}$$

Remarks on boundary conditions

$$\boldsymbol{F(d)}\ \boldsymbol{S(d)}\ \widehat{\boldsymbol{n}} = -k_s \boldsymbol{d} - c_s \frac{\partial \boldsymbol{d}}{\partial t} \quad \textit{("silent blocks")}$$

$$\boldsymbol{F(d)}\ \boldsymbol{S(d)}\ \widehat{\boldsymbol{n}} = p_0$$

Astorino, JFG

Remarks on boundary conditions

$$\boldsymbol{F}(\boldsymbol{d})\ \boldsymbol{S}(\boldsymbol{d})\ \widehat{\boldsymbol{n}} = -k_s\boldsymbol{d} - c_s\frac{\partial \boldsymbol{d}}{\partial t} \quad (\textit{"silent blocks"})$$

$$\boldsymbol{F}(\boldsymbol{d})\ \boldsymbol{S}(\boldsymbol{d})\ \widehat{\boldsymbol{n}} = p_0$$
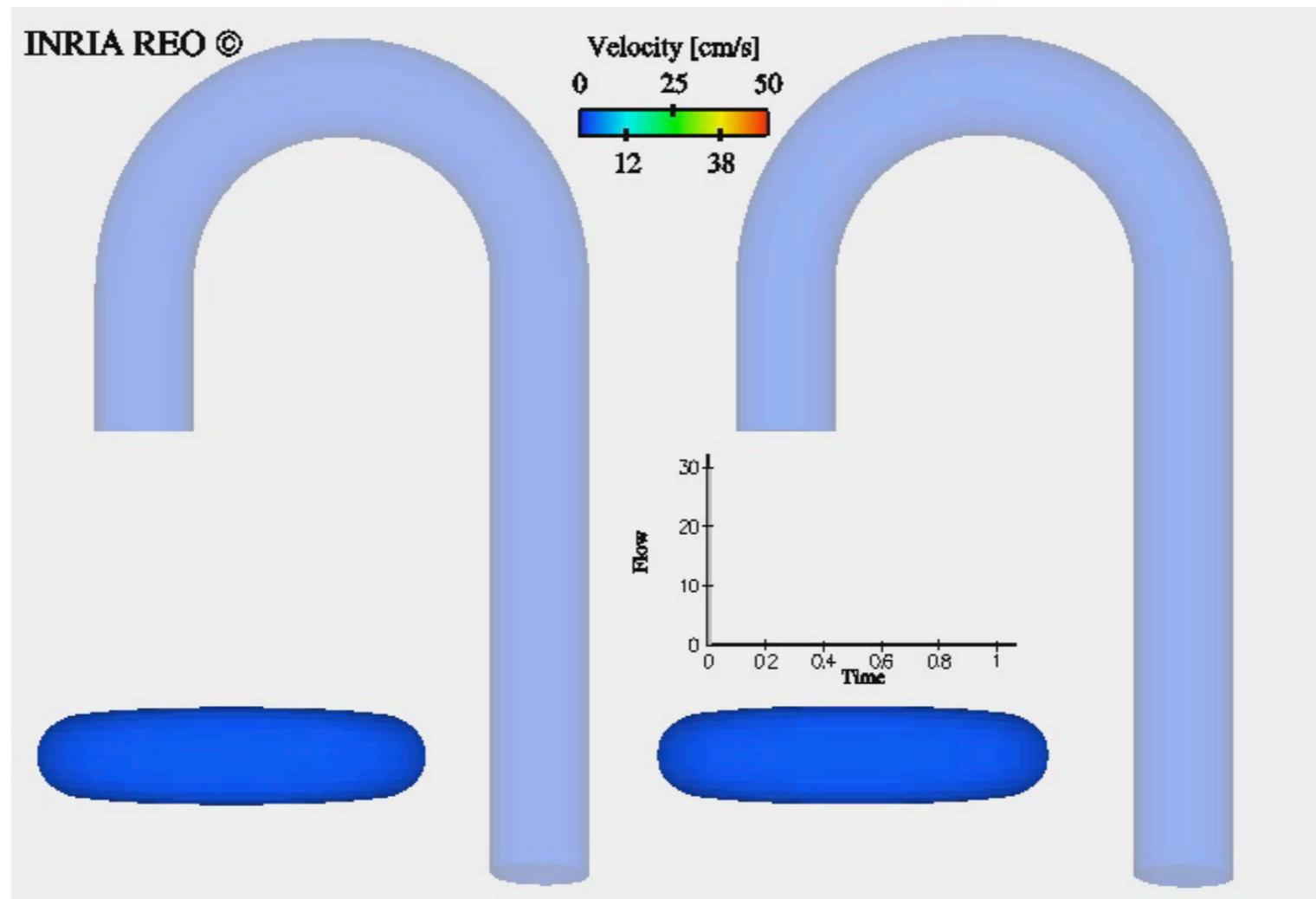
Astorino, JFG

Outline

- Fluid-structure interaction in arteries

    - Stability issues - Strongly couple schemes

    - A stable weakly coupled scheme

- Fluid-structure interaction with valves

    - Fictitious domain method
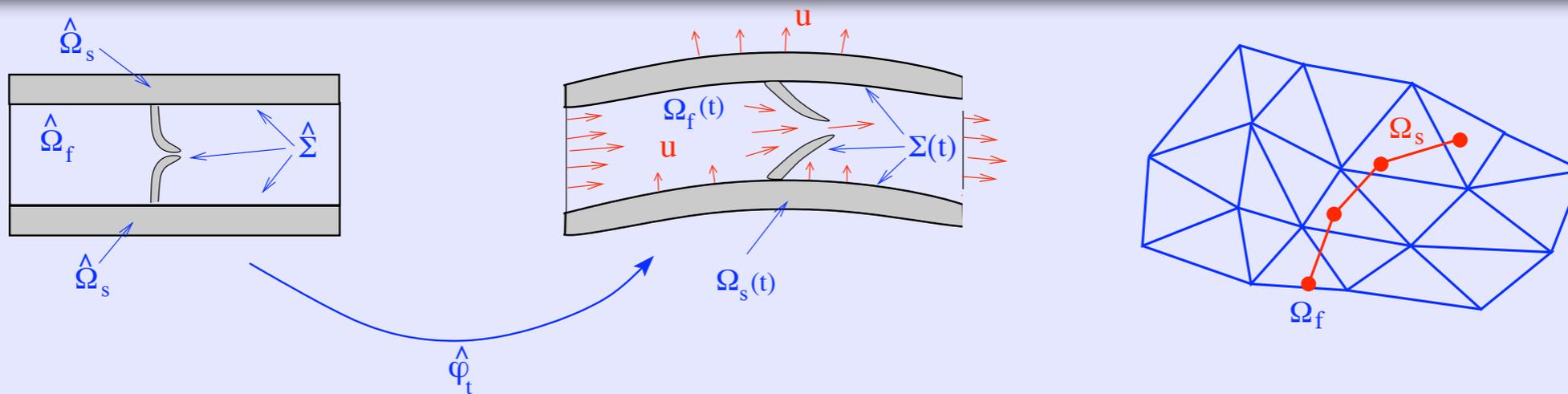
    - FSI and Contact management

FSI with valves

FSI with valves
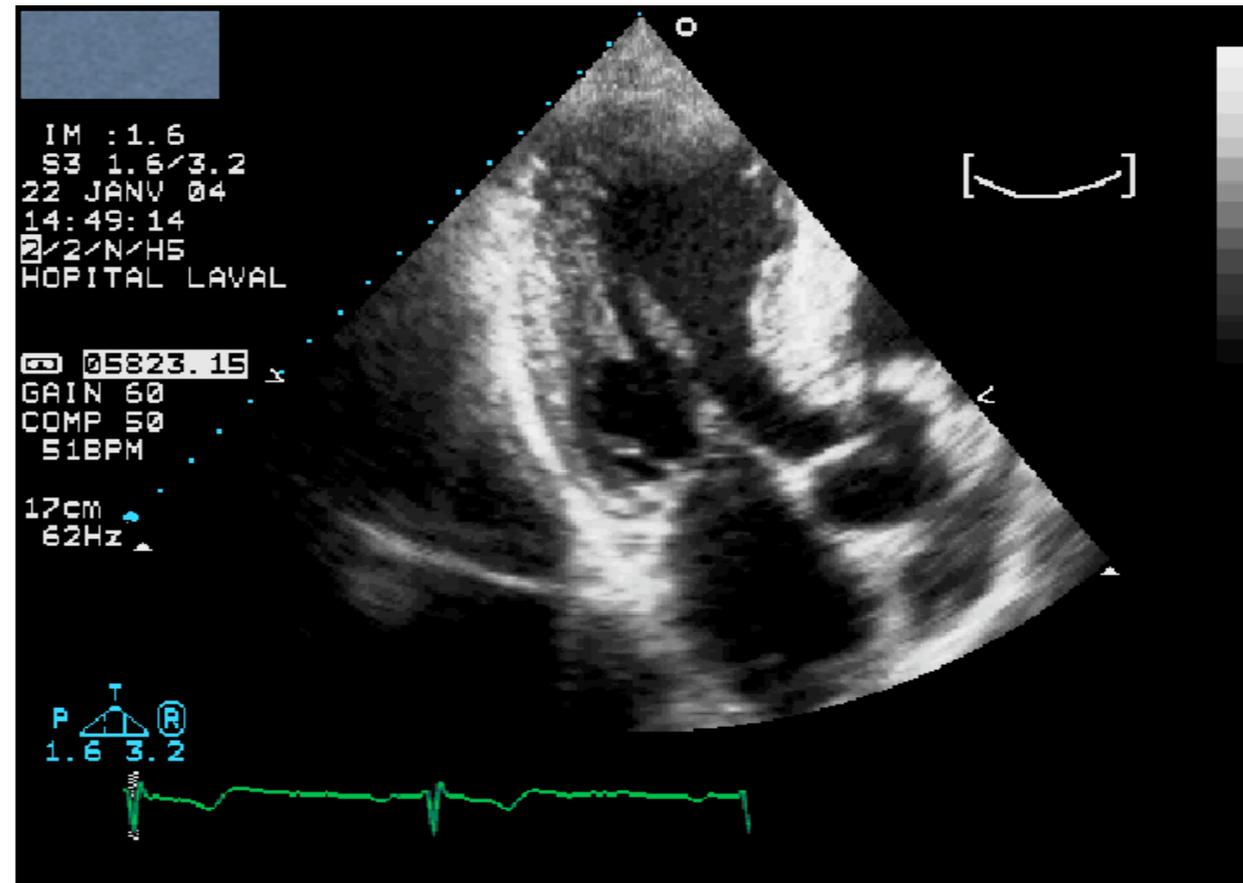
FSI with valves
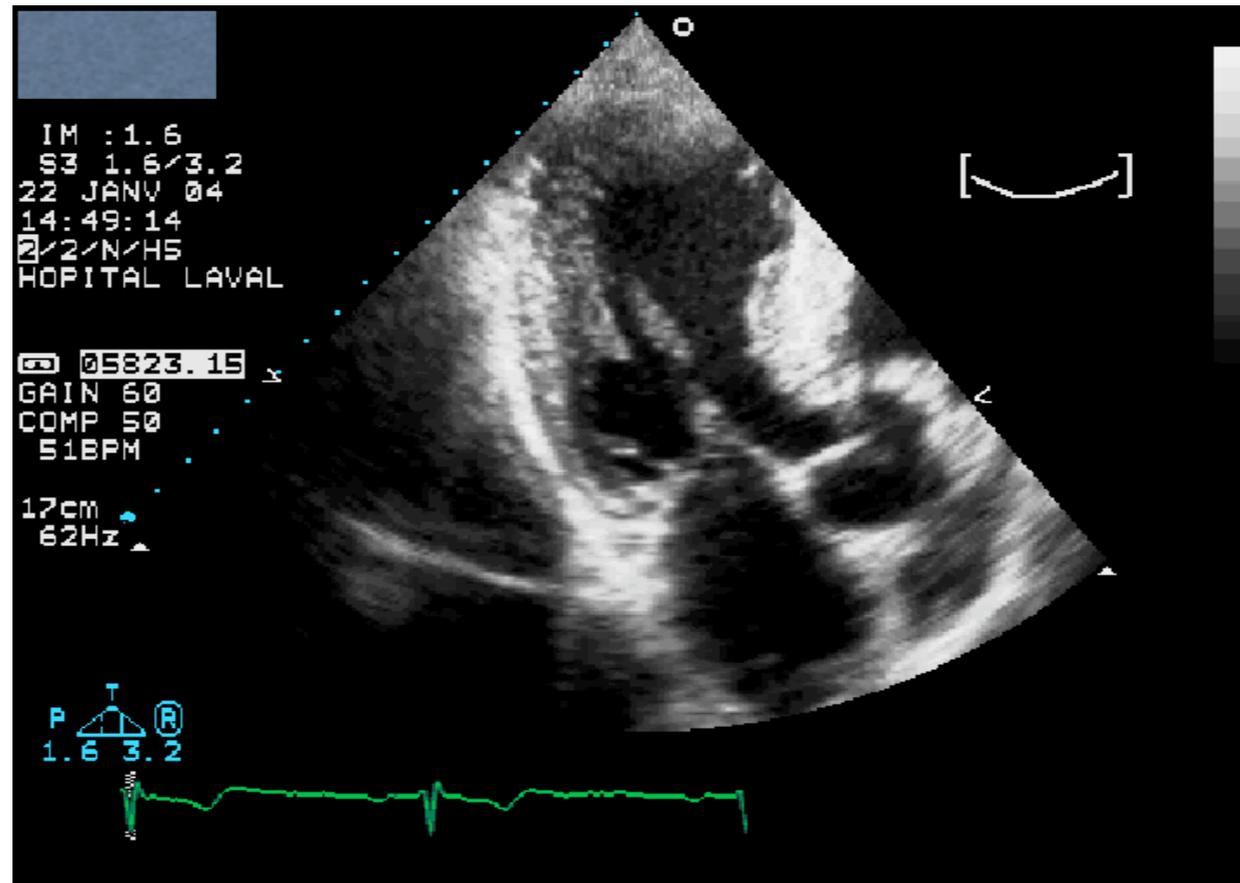
Valve : "Fictitious Domains" (FD)

"Fictitious Domain" for valves

- Basic idea: impose the kinematic constraint in a weak form

$$\langle \boldsymbol{\mu}, \boldsymbol{u}_f \rangle_\Sigma = \langle \boldsymbol{\mu}, \boldsymbol{u}_s \rangle_\Sigma, \forall \boldsymbol{\mu} \in \Lambda$$

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$$\langle \boldsymbol{\mu}, \boldsymbol{u}_f \rangle_\Sigma = \langle \boldsymbol{\mu}, \boldsymbol{u}_s \rangle_\Sigma, \forall \boldsymbol{\mu} \in \Lambda$$

- A saddle point problem has to be solved in the fluid

$$\begin{cases} a_f(\boldsymbol{u}_f, \boldsymbol{v}_f) + \langle \lambda, \boldsymbol{v}_f \rangle_\Sigma &= \displaystyle\int_{\Omega_f(t)} \boldsymbol{f}_f \cdot \boldsymbol{v}_f, \quad \forall \boldsymbol{v}_f \in X_f \\[2ex] \langle \mu, \boldsymbol{u}_f \rangle_\Sigma - \langle \mu, \boldsymbol{u}_s \rangle_\Sigma &= 0 \qquad\qquad \forall \mu \in \Lambda \\[2ex] \hat{a}_s(\hat{\boldsymbol{u}}_s, \hat{\boldsymbol{v}}_s) - \langle \lambda, \boldsymbol{v}_s \rangle_\Sigma &= \displaystyle\int_{\hat{\Omega}_s} \hat{\boldsymbol{f}}_s \cdot \hat{\boldsymbol{v}}_s \qquad \forall \hat{\boldsymbol{v}}_s \in \hat{X}_s \end{cases}$$

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$$
\begin{cases}
a_f(\boldsymbol{u}_f, \boldsymbol{v}_f) + \langle \lambda, \boldsymbol{v}_f \rangle_\Sigma & = & \displaystyle\int_{\Omega_f(t)} \boldsymbol{f}_f \cdot \boldsymbol{v}_f, & \forall \boldsymbol{v}_f \in X_f \\[4mm]
{\color{red}\langle \mu, \boldsymbol{u}_f \rangle_\Sigma - \langle \mu, \boldsymbol{u}_s \rangle_\Sigma} & {\color{red}=} & {\color{red}0} & {\color{red}\forall \mu \in \Lambda} \\[4mm]
\hat{a}_s(\hat{\boldsymbol{u}}_s, \hat{\boldsymbol{v}}_s) - \langle \lambda, \boldsymbol{v}_s \rangle_\Sigma & = & \displaystyle\int_{\hat{\Omega}_s} \hat{\boldsymbol{f}}_s \cdot \hat{\boldsymbol{v}}_s & \forall \hat{\boldsymbol{v}}_s \in \hat{X}_s
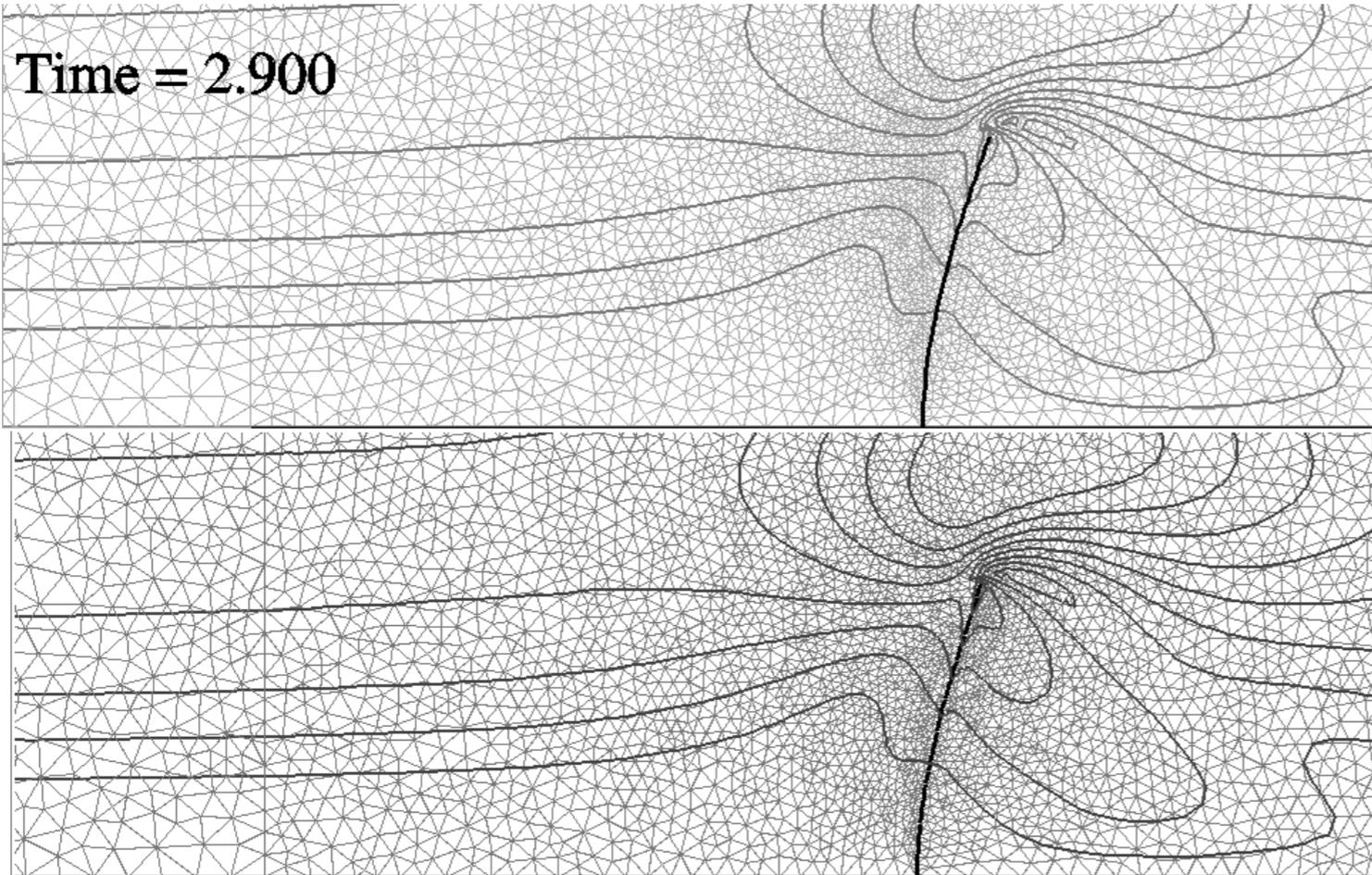\end{cases}
$$

- Lagrange multiplier space:

$$\Lambda_h = \{ \boldsymbol{\mu}_h \text{ measure on } \Sigma, \ \boldsymbol{\mu}_h = \sum_{i=1}^{N_\Sigma} \boldsymbol{\mu}_i \delta(\mathbf{x}_i^{n+1}), \boldsymbol{\mu}_i \in \mathbb{R}^n \}$$

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- Other Lagrange multiplier spaces are possible

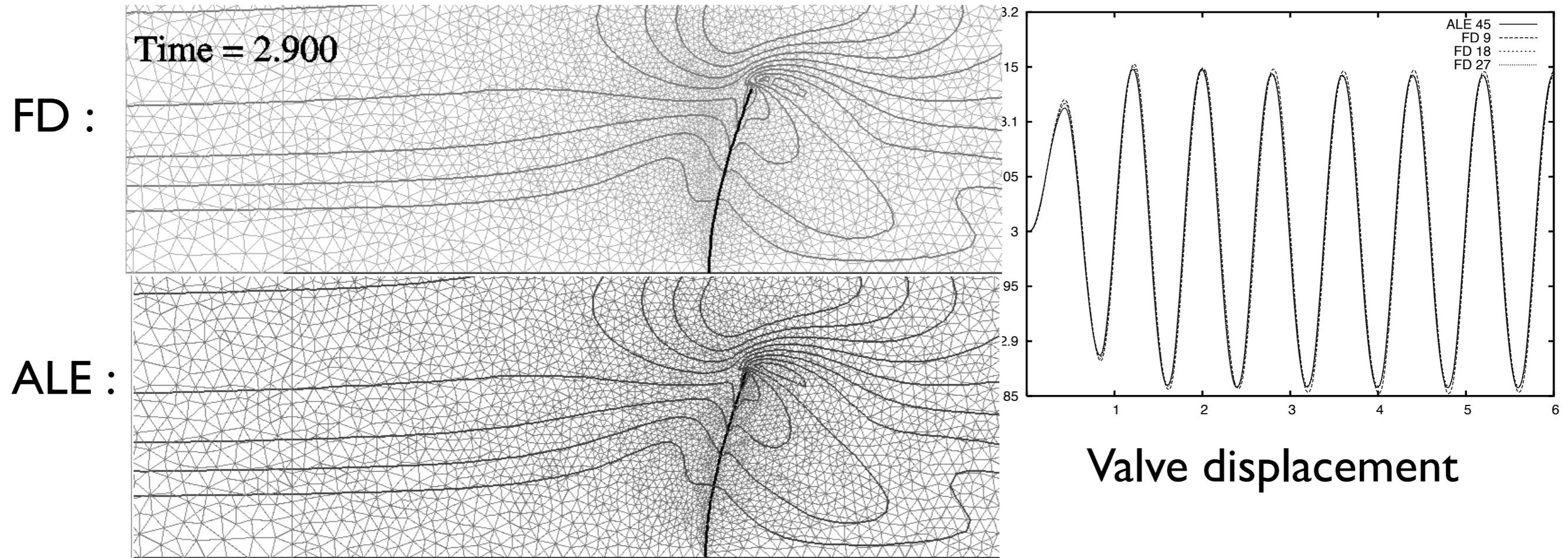*Baaijens, 2001, de Hart et al. 2003*
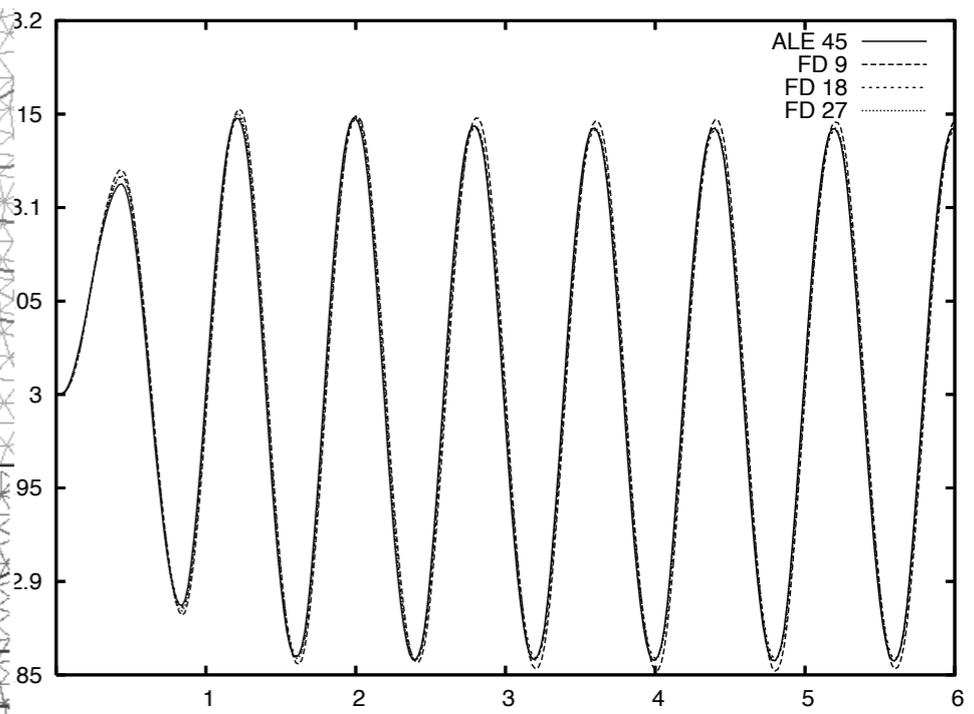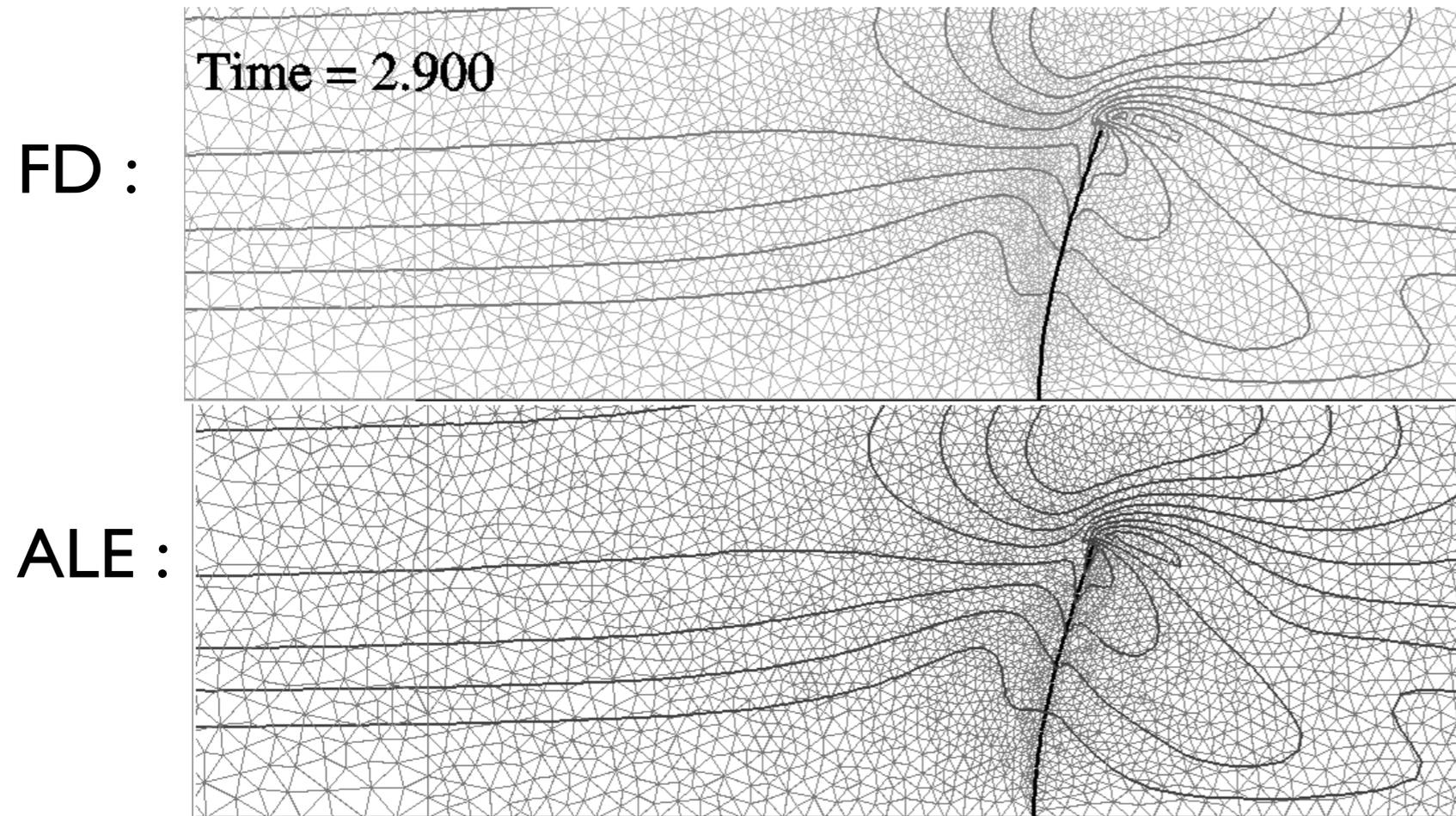
ALE / FD

Comparison:

FD :

ALE :

Time = 2.900

ALE / FD

Comparison:

FD :

ALE :

Valve displacement

ALE / FD

Comparison:

FD :

ALE :

Time = 2.900

Valve displacement

Mix ALE + FD:
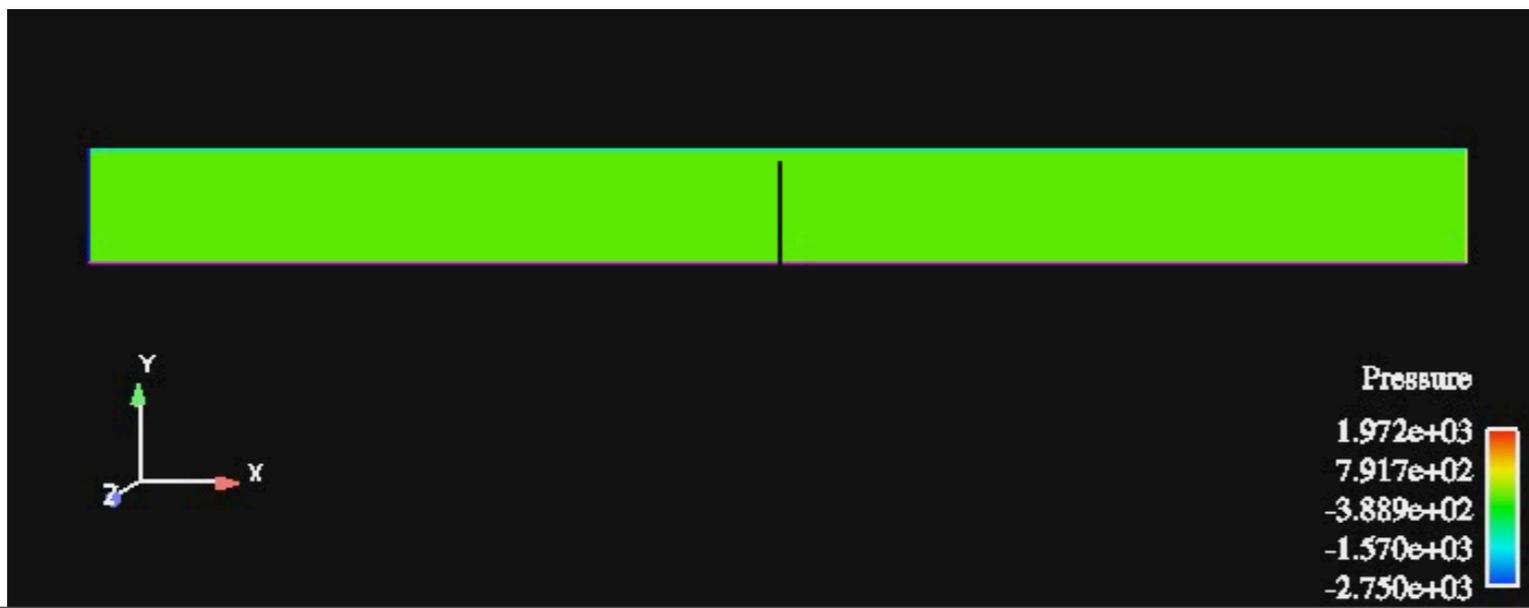
N. Diniz dos Santos

FSI and kinematic constraints

FSI and kinematic constraints

- Valves are submitted to various kinematic constraints:

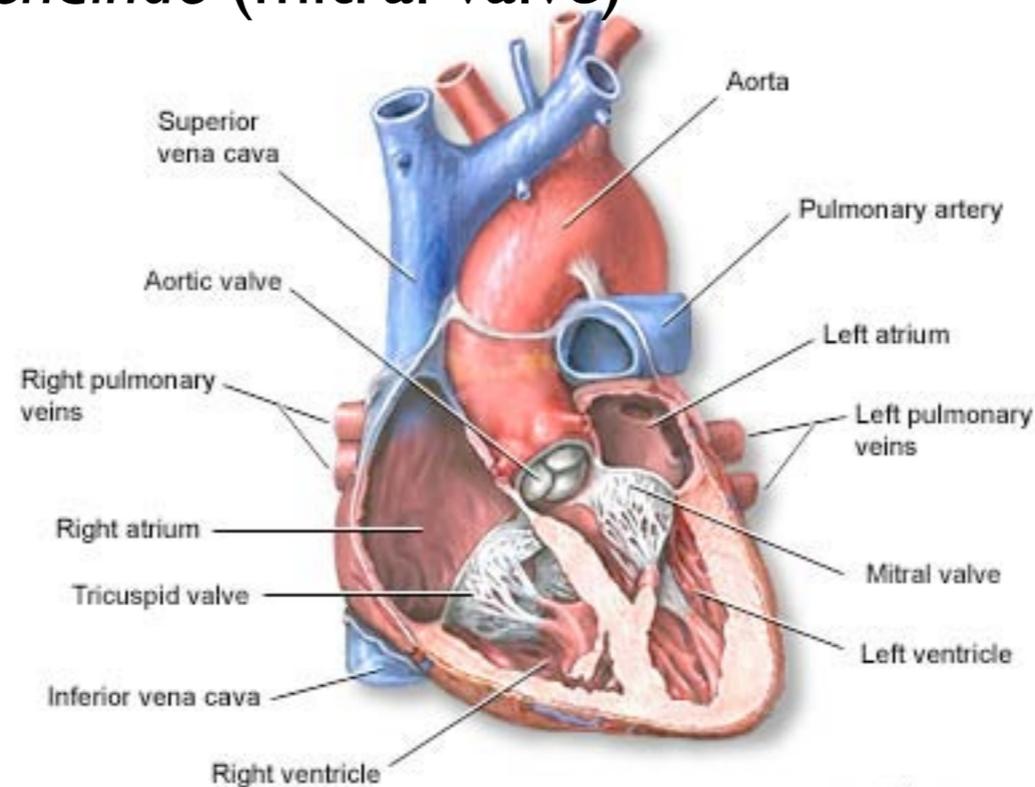# FSI and kinematic constraints

- Valves are submitted to various kinematic constraints:
 - contact between leaflets

FSI and kinematic constraints

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    - *chordae tendinae* (mitral valve)

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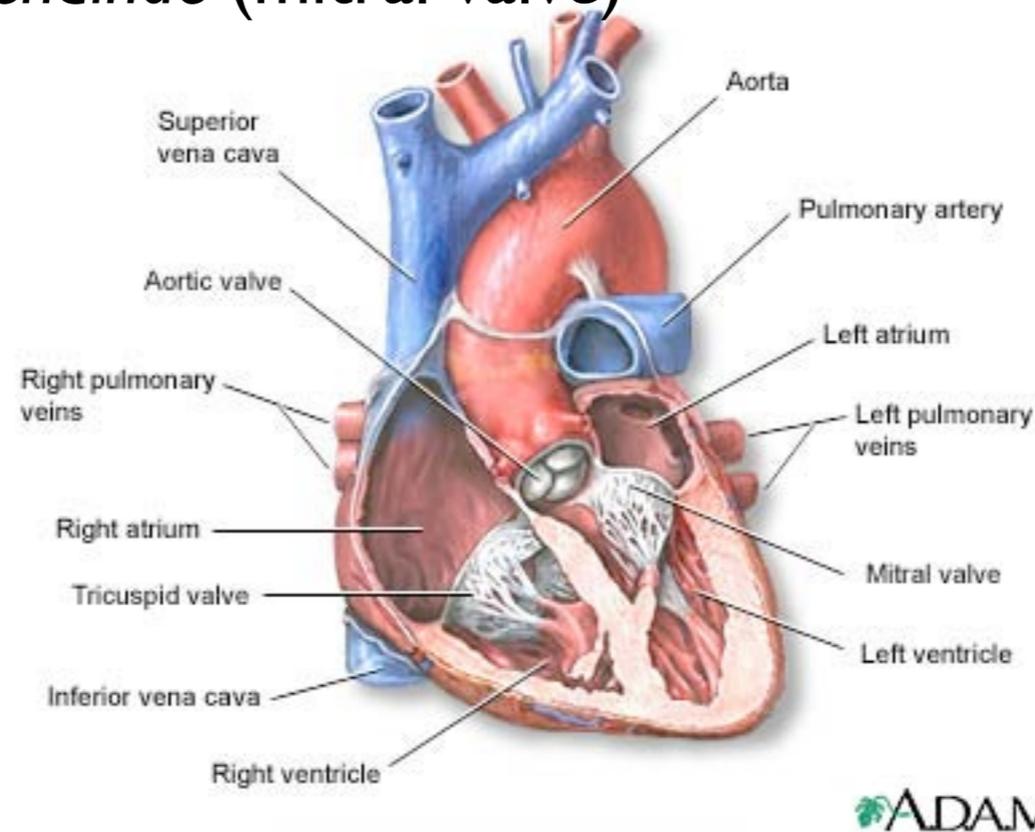
- We propose a framework to deal with these constraints in partitionned FSI algorithms.
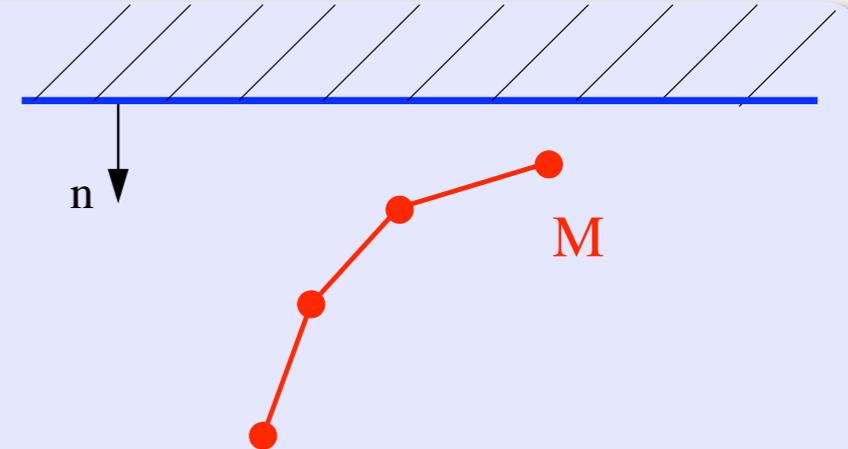
Solid-wall contact

M a solid with energy J

$\mathcal{T}_h$ a mesh of M

$X_h = \{\varphi_h \in C^0(M; \mathbb{R}^3), \varphi_h|_T \in P_1, \forall T \in \mathcal{T}_h\}$

Mininization with convex constraint : $\displaystyle\inf_{\varphi_h \in \mathcal{U}_h} J(\varphi_h)$

Solid-wall contact
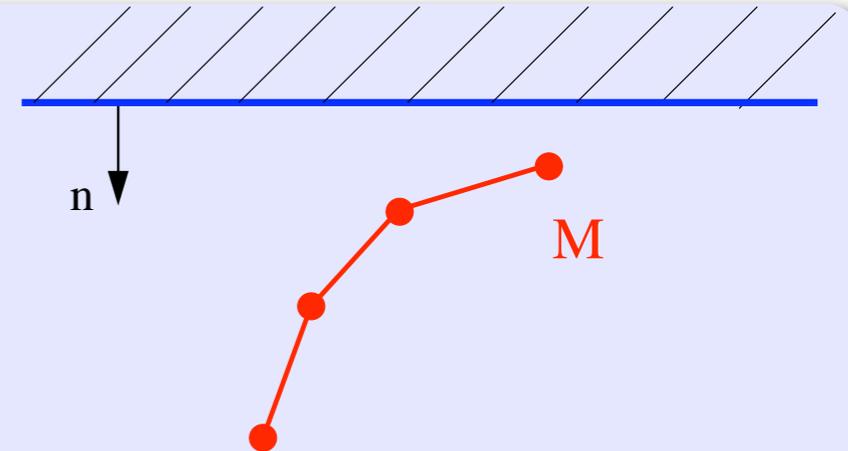

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With $\mathcal{U}_h = \{\varphi_h \in X_h, F_{x_i}(\varphi_h) \leq 0, \forall x_i \in M\}$

$F_{x_i}(\boldsymbol{\varphi}_h) = \varepsilon - \boldsymbol{n} \cdot \boldsymbol{\varphi}_h(x_i) - c$

Solid-wall contact
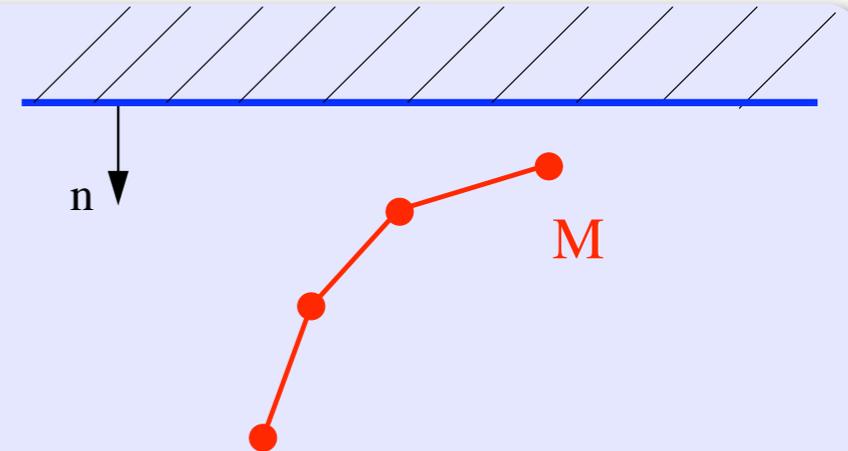
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van Loon, Anderson, van de Vosse, 2006

Dual approach

$$G(\mu) = \inf_{\boldsymbol{\varphi} \in X_h} \left[J(\boldsymbol{\varphi}) + \sum_{i=1}^{N_\Sigma} \mu_i F_{x_i}(\boldsymbol{\varphi}) \right]$$

$$G(\lambda_c) = \max_{\mu_i \geq 0} G(\mu)$$

λ_c : contact pressure

Dual approach

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- The constraint is now simple to enforce

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Gradient method with projection

1) $\langle J'(\boldsymbol{\varphi}^k), \boldsymbol{\xi} \rangle = -\sum_{i=1}^{N_\Sigma} \lambda_{c,i}^k \langle F'_{x_i}(\boldsymbol{\varphi}^k), \xi \rangle = \sum_{i=1}^{N_\Sigma} \lambda_{c,i}^k \boldsymbol{n} \cdot \boldsymbol{\xi}(x_i)$

2) $\lambda_{c,i}^{k+1} = \mathbf{P}_{\mathbb{R}^+} \left(\lambda_{c,i}^k + \alpha^k \boldsymbol{\nabla} G(\lambda_c^k)_i \right) = \mathbf{P}_{\mathbb{R}^+} \left(\lambda_{c,i}^k + \alpha^k F_{x_i}(\boldsymbol{\varphi}^k) \right)$

3) Iterate on k

Dual approach

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🔘 The constraint is now simple to enforce

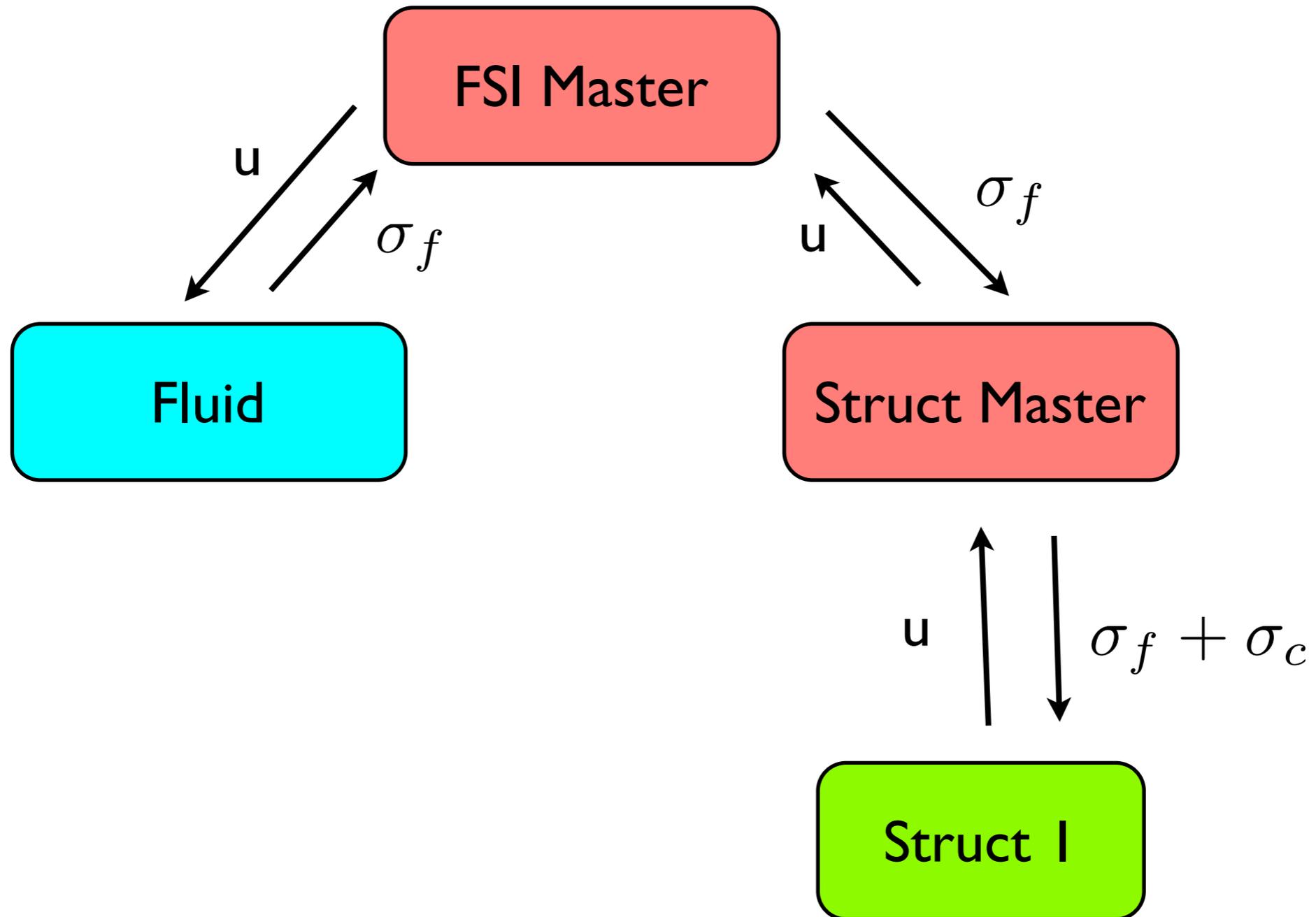## Gradient method with projection

1) $\langle J'(\boldsymbol{\varphi}^k), \boldsymbol{\xi} \rangle = -\sum_{i=1}^{N_\Sigma} \lambda_{c,i}^k \langle F'_{x_i}(\boldsymbol{\varphi}^k), \xi \rangle = \sum_{i=1}^{N_\Sigma} \lambda_{c,i}^k \boldsymbol{n} \cdot \boldsymbol{\xi}(x_i)$
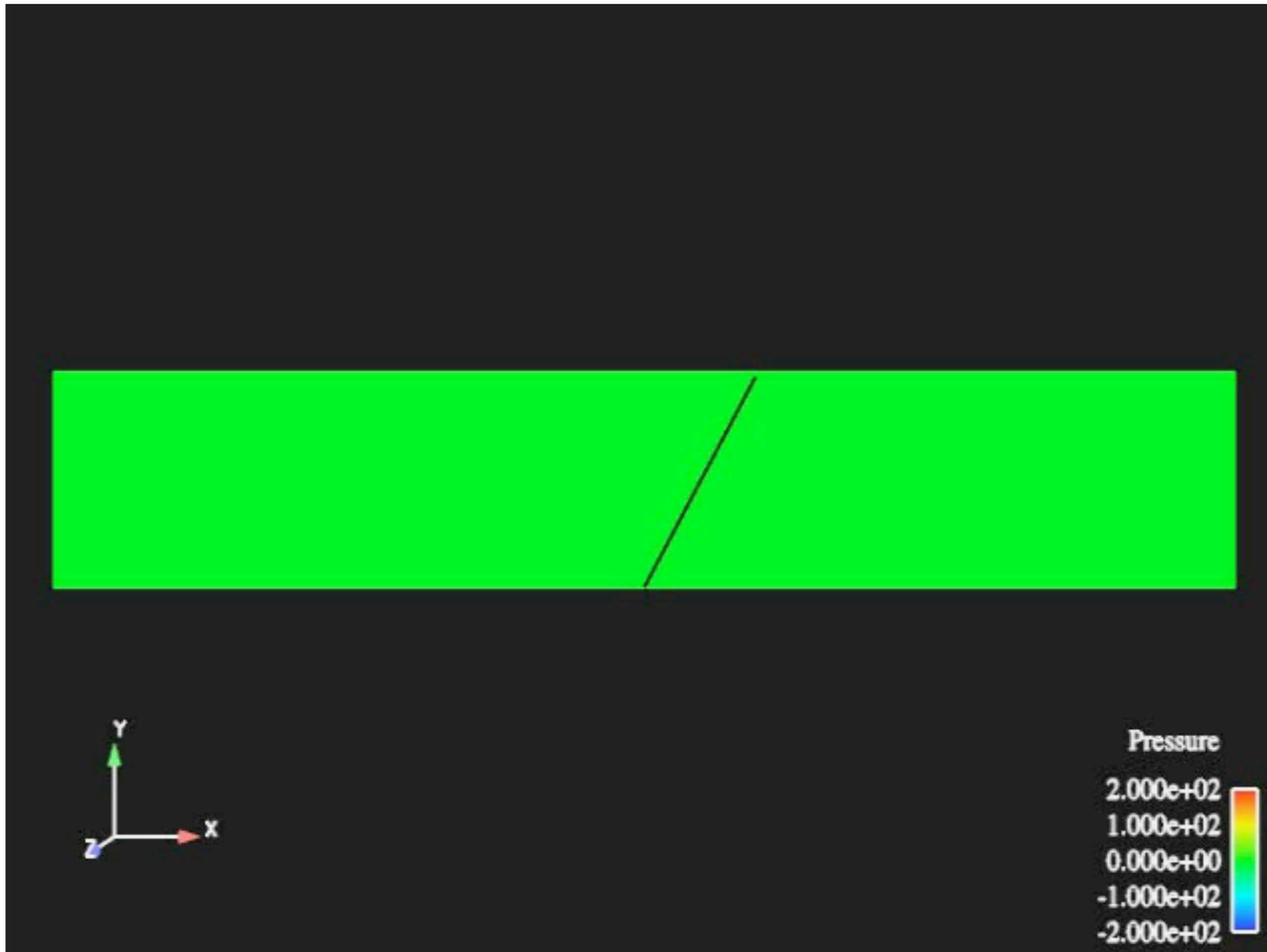
2) $\lambda_{c,i}^{k+1} = \mathbf{P}_{\mathbb{R}^+} \left(\lambda_{c,i}^k + \alpha^k \boldsymbol{\nabla} G(\lambda_c^k)_i \right) = \mathbf{P}_{\mathbb{R}^+} \left(\lambda_{c,i}^k + \alpha^k F_{x_i}(\boldsymbol{\varphi}^k) \right)$

3) Iterate on k

🔘 The contact force is added to the hydrodynamics force

Implementation

Diniz dos Santos, JFG, Bourgat 2007
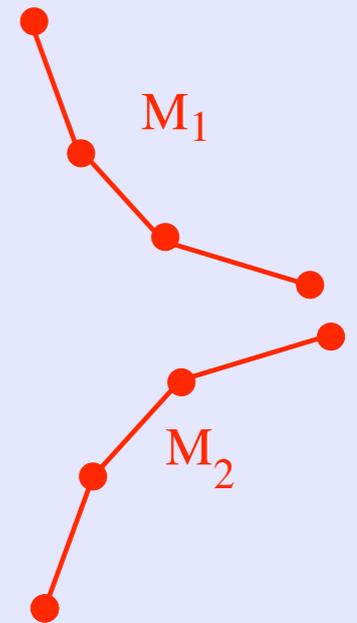
Solid-solid contact

$M = (M_1, M_2, \dots)$ a family of solid with energy J

$\mathcal{T}_h$ a mesh of M

$X_h = \{\varphi_h \in C^0(M; \mathbb{R}^3), \varphi_h|_T \in P_1, \forall T \in \mathcal{T}_h\}$

Mininization with *non convex* constraints :

$$\inf_{\varphi_h \in \mathcal{U}_h} J(\varphi_h)$$

M_1

M_2

Solid-solid contact

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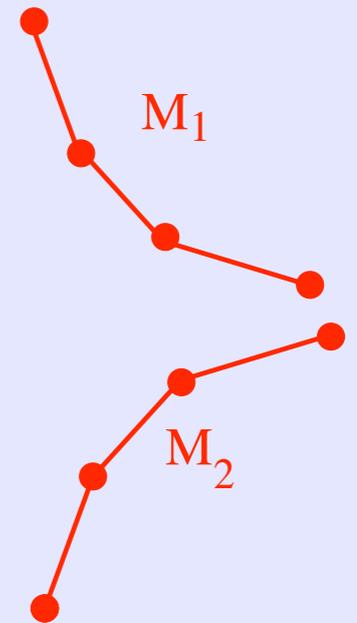
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Mininization with *non convex* constraints :

$$\inf_{\varphi_h \in \mathcal{U}_h} J(\varphi_h)$$

with

$\mathcal{U}_h = \{\boldsymbol{\varphi}_h \in X_h, dist(\boldsymbol{\varphi}_h(T_1), \boldsymbol{\varphi}_h(T_2)) \geq \varepsilon, \forall T_1, T_2 \in \mathcal{T}_h\}$
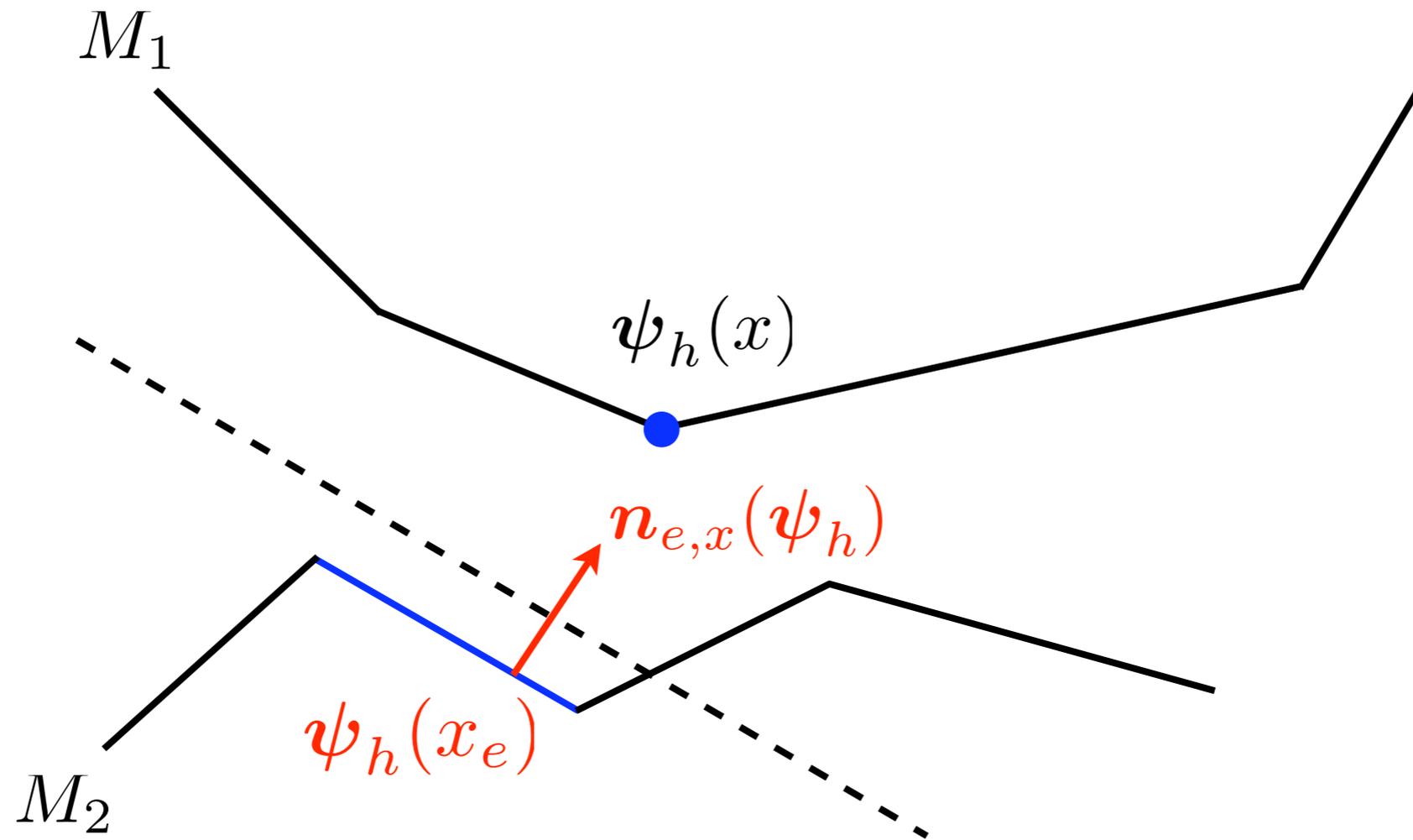
Solid-solid contact

Optimization algorithm (O. Pantz, 2007) :

- replace a problem with nonconvex constraints with a sequence of problems with convex constraints.

- Solve
$$J(\boldsymbol{\varphi}_h^{k+1}) = \inf_{\boldsymbol{\psi}_h \in T(\boldsymbol{\varphi}_h^k)} J(\boldsymbol{\psi}_h)$$

  where $T(\boldsymbol{\varphi}_h^k)$ is a convex neighborhood of $\boldsymbol{\varphi}_h^k$

- Iterate on k until convergence

Solid-solid contact

$$T(\boldsymbol{\psi}_h) = \Big\{ \boldsymbol{\varphi}_h \in X_h, \min_{x_e \in e} \boldsymbol{n}_{e,x}(\boldsymbol{\psi}_h) \cdot (\boldsymbol{\varphi}_h(x_e) - \boldsymbol{\varphi}_h(x)) \geq \varepsilon$$

$$\text{for all edges } e \text{ and all node } x \notin e \Big\}$$

- At convergence, φ_h does not satisfy *a priori* the optimality conditions of the original problem

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- ... but the error is O(h) (*O. Pantz, 2007*)

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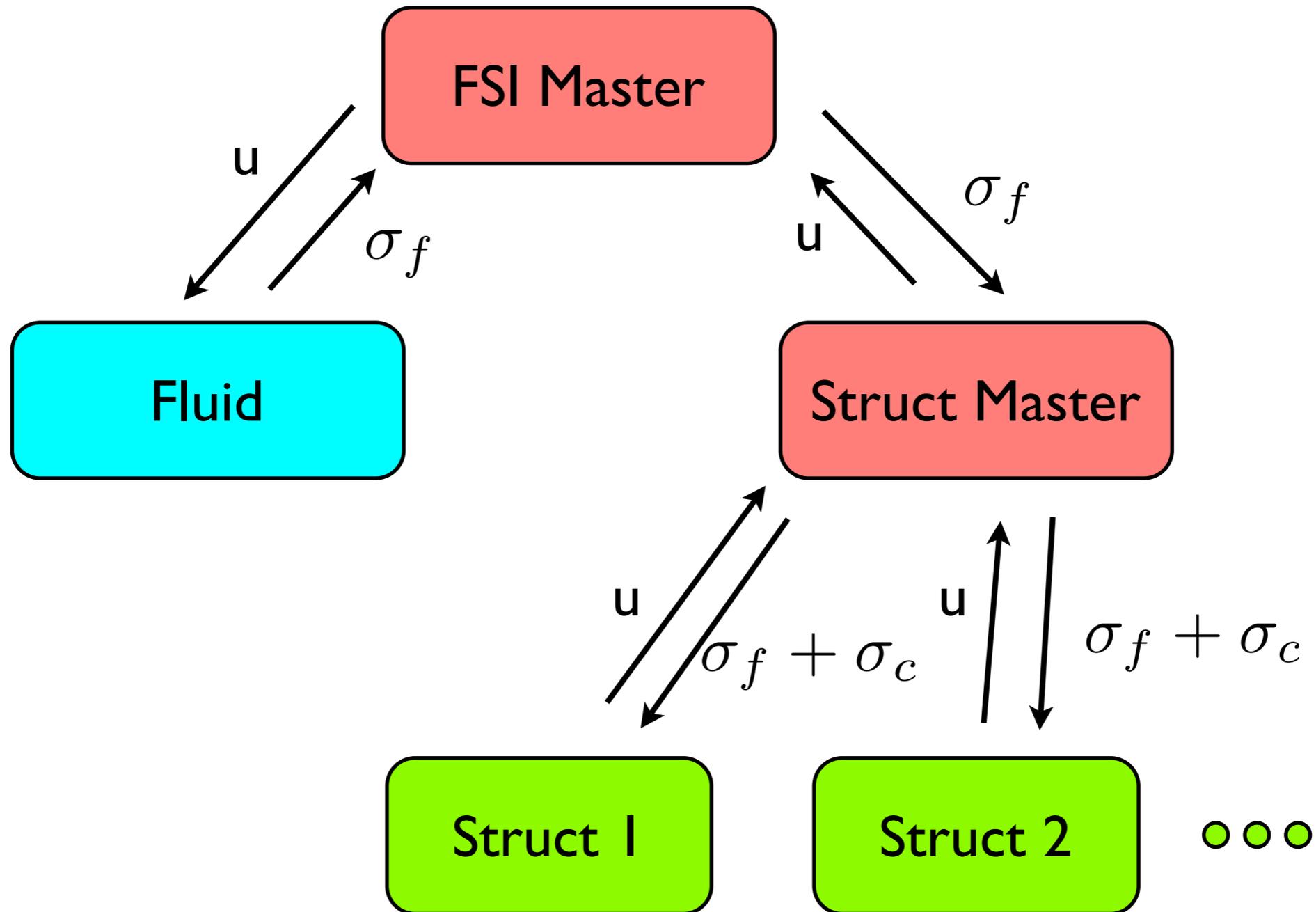- Same kinds of constraint as for the solid-wall case:

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- ... but the error is O(h) (*O. Pantz, 2007*)

- Same kinds of constraint as for the solid-wall case:

$$T(\boldsymbol{\psi}_h) = \Big\{ \boldsymbol{\varphi}_h \in X_h, F^0_{e,x}(\boldsymbol{\varphi}_h) \leq 0, F^1_{e,x}(\boldsymbol{\varphi}_h) \leq 0,$$

$$\text{for all edges } e \text{ and all node } x \notin e \Big\}$$

$$F^j_{e,x}(\boldsymbol{\varphi}_h) = \varepsilon - \boldsymbol{n}_{e,x}(\boldsymbol{\psi}_h) \cdot (\boldsymbol{\varphi}_h(e_j) - \boldsymbol{\varphi}_h(x))$$

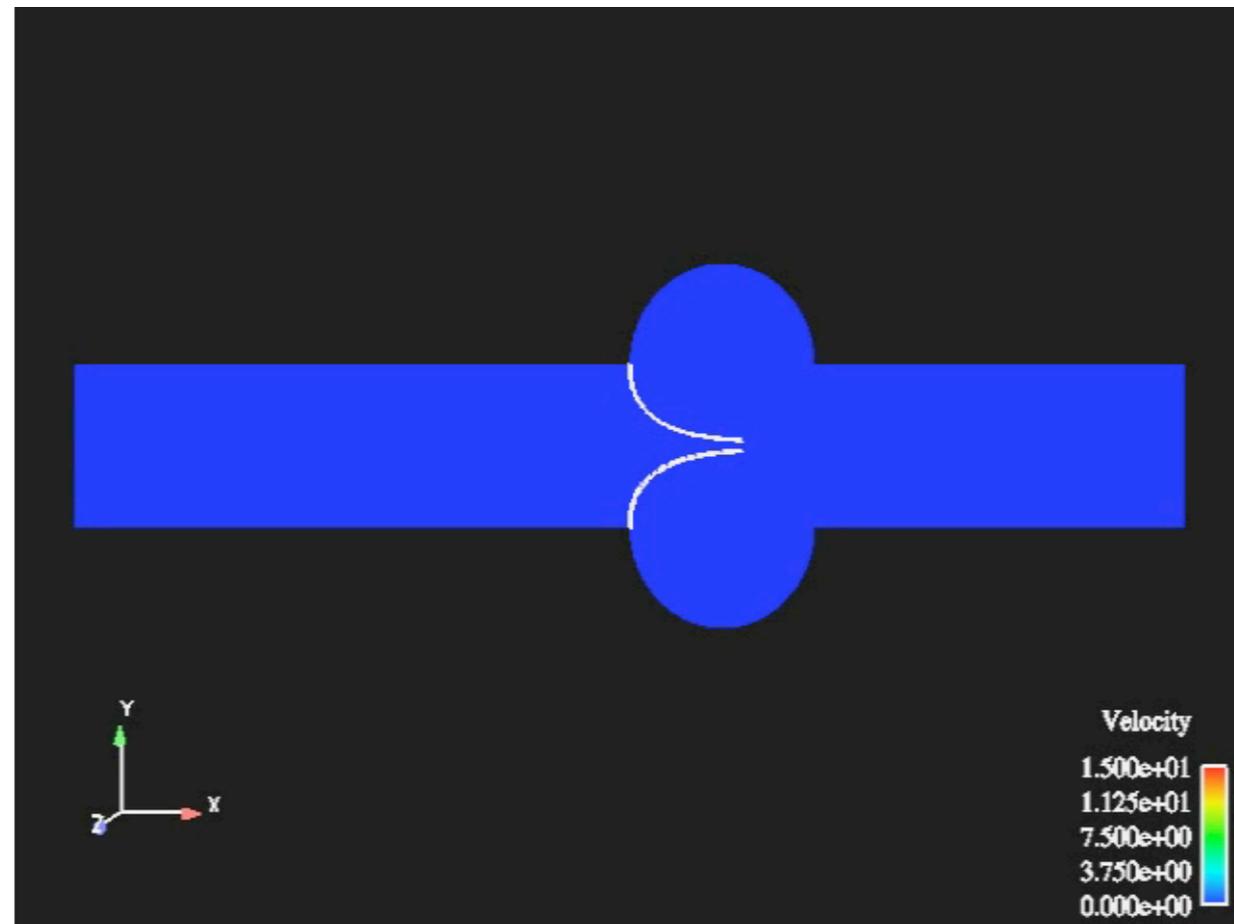# Implementation

FSI & Solid-solid contact

Diniz dos Santos, JFG, 2007

FSI & Solid-solid contact

Diniz dos Santos, JFG, 2007

Time = 0.00000

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Velocity

1.119e+01
8.392e+00
5.595e+00
2.797e+00
0.000e+00
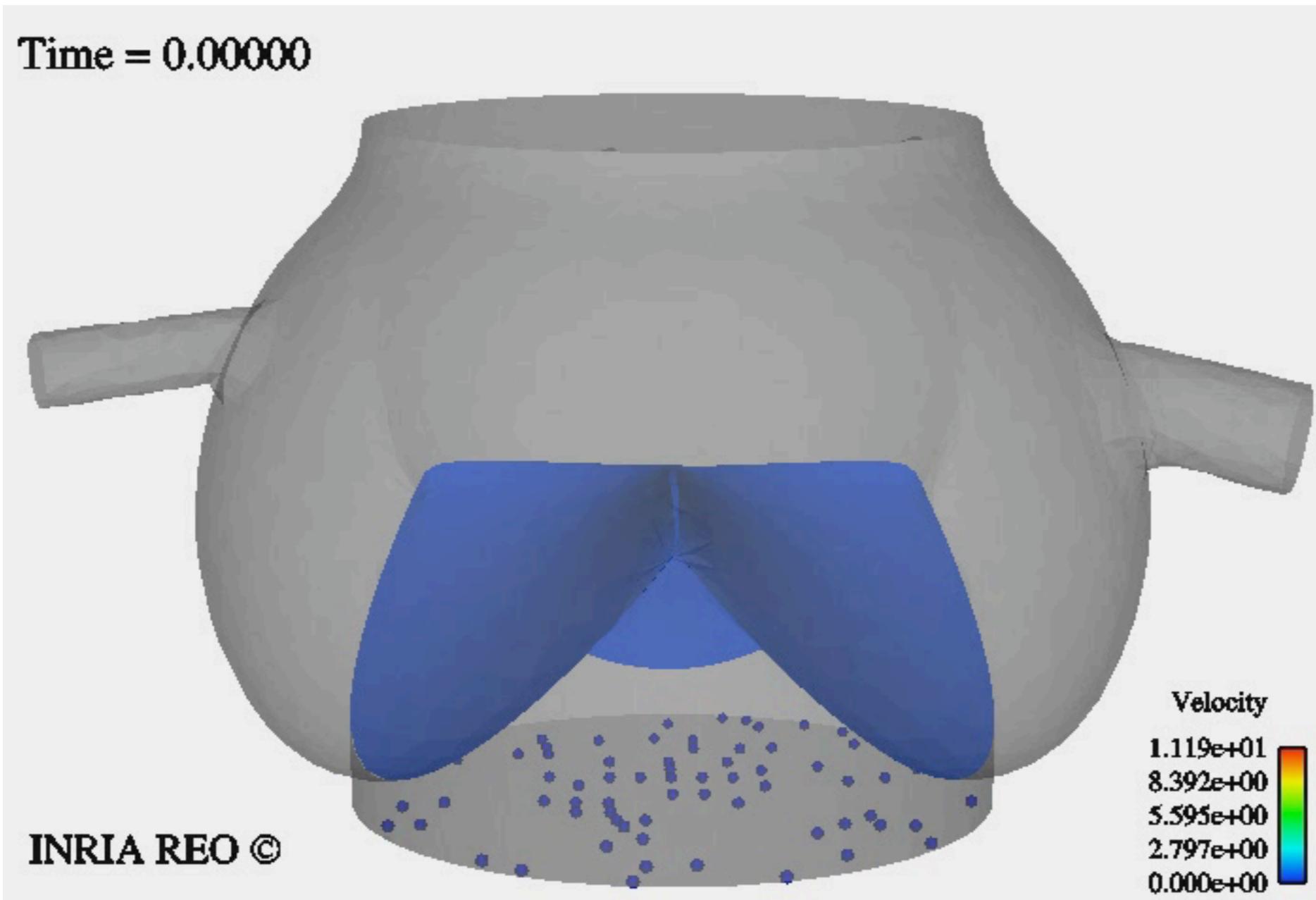
Astorino, JFG, Pantz, Traoré, 2008

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- Grand challenge: coupling numerical models with medical data.

Collaborators

- M. Astorino (INRIA)
- N. Diniz dos Santos (INRIA)
- M.A. Fernandez (INRIA)
- C. Grandmont (INRIA)
- V. Martin (UTC)
- O. Pantz (Ecole Polytechnique)