A new numerical method for the valuation of a non callable defaultable coupon bond under an extended JDCEV model

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Abstract

In this paper, we consider the numerical solution of a two factor model for the valuation of a non callable defaultable bond which pays coupons at certain given dates. Such a model under consideration is the Jump to Default Constant Elasticity of Variance (JDCEV) model. The stochastic factors are the risk free interest rate and the stock price. Moreover, we take into account the possibility of negative interest rates. From the mathematical point of view, the valuation problem requires the numerical solution of two partial differential equation (PDE) problems for each coupon and with maturity those coupon payment dates. In order to solve these PDE problems, we propose appropriate numerical schemes based on a Crank-Nicolson semi-Lagrangian method for time discretization combined with biquadratic Lagrange finite elements for space discretization. Once the numerical solutions of the PDEs are obtained, a kind of post-processing is carried out in order to achieve the value of the bond. This post-processing includes the computation of an integral term which is approximated by using the composite trapezoidal rule. Finally, we present some numerical results for real market bonds issued by different firms in order to illustrate the proper behaviour of the numerical schemes.

Keywords: coupon bond, default risk, JDCEV model, PDE, semi-Lagrangian method, biquadratic Lagrange finite elements.

1. Introduction

Default risk can be modelled by using two different approaches: the structural approach and the reduced-form approach. For more information about the history, advantages and drawbacks of each one of the two approaches we refer the readers to the introduction of [1, 8, 9]. In the structural method default occurs when the firm asset value reaches some lower barrier, whereas in the reduced-form approach, the default event is assumed to be unpredictable and governed by a default intensity process that could be either deterministic or stochastic. Moreover, in the reduced-form approach the default event can occur without any correlation with the firm value. In this work we consider the reduced-form approach.

In the literature there are several papers devoted to default risk modelling by using the two different approaches. On one hand, the first approach is taking into consideration in [1] where the author proves an exact formula for the valuation of defaultable coupon bonds by generalizing the one derived in [13]. On the other hand, both approaches are combined in [9] also to price defaultable bonds.

The main objective of this paper is to obtain the price of a defaultable coupon bond under the extended Jump to Default Constant Elasticity of Variance (JDCEV) model proposed in [7], also considering the possibility of incorporating negative interest rates which is introduced in [11]. The JDCEV model was first introduced in [7] as a hybrid credit and equity model, although only taking into account

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constant positive interest rates. In this work we assume that the interest rates dynamics is governed by a stochastic process which takes into account negative interest rates as in [11]. The incorporation of negative interest rates results more realistic according to the current situation of real markets. As it was pointed out before, this model can be set in the framework of the reduced-form approaches. More precisely, in the present paper we define the instantaneous volatility of the stock price as a constant elasticity of variance (CEV) process and we assume that the default intensity is an affine function of the instantaneous variance of the underlying stock.

From the mathematical point of view, the valuation problem of a defaultable coupon bond can be posed in terms of a sequence of partial differential equation (PDE) problems, where the underlying stochastic factors are the interest rates and the stock price. Moreover, the stock price follows a diffusion process interrupted by a possible jump to zero (default), as it is indicated in [7]. In order to compute the value of the bond we need to solve two partial differential equation problems for each coupon and with maturity those coupon payment dates. Concerning the numerical solution of those PDE problems, after a localization procedure to formulate the problems in a bounded domain and the study of the boundaries where boundary conditions are required following the ideas introduced in [15], we propose appropriate numerical schemes based on a Crank-Nicolson semi-Lagrangian method for time discretization combined with biquadratic Lagrange finite elements for space discretization. The numerical analysis of this Lagrange-Galerkin method has been addressed in [3, 4]. Once the numerical solution of the PDEs is obtained, a kind of post-processing is carried out in order to achieve the value of the bond. This post-processing includes the computation of an integral term which is approximated by using the composite trapezoidal rule.

This paper is organized as follows. In Section 2, we describe the two stochastic factors (i.e. the interest rate and the defaultable stock price) that are involved in the model, and we state the PDE problem that governs the valuation of non callable defaultable coupon bonds. In Section 4, we formulate the pricing problem in a bounded domain after a localization procedure and we impose appropriate boundary conditions. Then, we introduce the discretization in time of the problem by using a Crank-Nicolson characteristic scheme, and we state the variational formulation of the problem in order to apply finite elements. Finally, in Section 5 we present some numerical results to illustrate the good performance of the numerical methods and we finish with some conclusions in Section 6.

2. Mathematical modelling

2.1. Stochastic underlying variables

A non callable defaultable coupon bond is a financial derivative product, the underlying variables of which are the interest rate and the defaultable stock price. The interest rate at time t, r_t, is assumed to be stochastic and its dynamics under a risk neutral probability measure is driven by the Vasicek model [17], in which the spot rate is governed by the Ornstein-Uhlenbeck process:

$$dr_t = \kappa(\theta - r_t)d_t + \delta \, dW_t^1, \tag{1}$$

where $\kappa > 0$ is the speed of adjustment in the mean reverting process, $\theta > 0$ is the long-term mean of the short-term interest rate, $\delta > 0$ is the interest rate volatility and W_t^1 is the standarized Wiener process for the interest rate. Note that negative interest rates are taken into consideration with this model contrary to other interest rate dynamics such as CIR model [10]. Note that the possibility of negative interest rates results more realistic according to the current situation of the markets.

In the JDCEV model we assume that the local volatility of the stock is given by

$$\sigma(t, S_t) = a(t) \, S_t^{\beta},$$

where S_t denotes the stock price at time t, $\beta < 0$ is the elasticity parameter and $a(t) > 0$ is the time dependent volatility scale function. Moreover, the default intensity can be written in terms of the stock volatility and the stock price in the following way

$$\lambda(t, S_t) = b(t) + c \, \sigma(t, S_t)^2 = b(t) + c \, a(t)^2 \, S_t^{2\beta}, \tag{2}$$

where $b(t) \geq 0$ is a deterministic non-negative function of time and $c > 0$ governs the sensitivity of the default intensity with respect to the volatility, as it is indicated in [7]. The other source of uncertainty, the predefaultable stock price at time t, S_t, under a risk neutral probability measure satisfies the following stochastic differential equation:

$$dS_t = (r_t + \lambda(t, S_t))S_t d_t + \sigma(t, S_t) S_t dW_t^2. \tag{3}$$

where W_t^2 is the standarized Wiener process for the stock price. Both Wiener processes, W_t^1 and W_t^2 can be correlated according to $dW_t^1 dW_t^2 = \rho dt$, where ρ denotes the instantaneous correlation coefficient such that $|\rho| < 1$. Moreover, in this model the time of default denoted by ξ has two parts. On one hand, we can define the predictable part as $\xi_0 = \inf\{t \geq 0 : S_t = 0\}$, whereas on the other hand the random part $\tilde{\xi}$ is given by the following expresion:

$$\tilde{\xi} = \inf\left\{t \geq 0 : \int_0^t \lambda(t, S_t) \geq e\right\}, \tag{4}$$

where e is an exponential random variable $e \sim Exp(1)$. Thus, the time of default is defined as $\xi = \xi_0 \wedge \tilde{\xi}$.

Finally, the defautable stock price is given by $\overline{S}_t = \{S_t \mathbb{1}_{\{\xi > t\}}\}$.

2.2. PDE formulation

In this work we consider the valuation of a non callable defaultable bond with the possibility of paying a series of coupons at given dates t_i, for $i = 1, ..., M$, where M is the number of coupons and $T = t_M$ is the maturity of the bond. At maturity, the bond holder receives the face value (FV) plus the last coupon. Thus, let us denote by cp_i the amount of money paid at coupon payment date t_i. This amount is computed by multiplying the coupon rate, the frequency of coupon payments and the FV. Having this in view, the value of a non callable defaultable coupon bearing bond at time $t = t_0 = 0$, $V(0, S_t, r_t; T)$, understood as the value of the future coupon payments and the face value of the bond, is given by

$$V(0, S_t, r_t; T) = FV\left[\sum_{i=1}^M cp_i \, \mathbb{E}\left[\exp\left(-\int_0^{t_i} (r_u + \lambda(u, S_u)) \, du\right)\right] + \mathbb{E}\left[\exp\left(-\int_0^T (r_u + \lambda(u, S_u)) \, du\right)\right]\right.$$
$$\left. +\eta\left(1 - \mathbb{E}\left[\exp\left(-\int_0^T (r_u + \lambda(u, S_u)) \, du\right)\right] - \int_0^T \mathbb{E}\left[\exp\left(-\int_0^{\tau_1} (r_u + \lambda(u, S_u)) \, du\right) r_{\tau_1}\right] d\tau_1\right)\right], \tag{5}$$

where η is the recovery rate in case of default. An analysis of the recovery rate by industrial sector and debt seniority is carried out in [2].

Next, if we denote by

$$u_1(0, S_t, r_t; t_i) = \mathbb{E}\left[\exp\left(-\int_0^{t_i} (r_u + \lambda(u, S_u)) \, du\right)\right],$$

$$u_1(0, S_t, r_t; T) = \mathbb{E}\left[\exp\left(-\int_0^T (r_u + \lambda(u, S_u)) \, du\right)\right],$$

$$u_2(0, S_t, r_t; \tau_1) = \mathbb{E}\left[\exp\left(-\int_0^{\tau_1} (r_u + \lambda(u, S_u)) \, du\right) r_{\tau_1}\right],$$

the expression of the bond value (5) can be written equivalently as

$$V(0, S_t, r_t; T) = FV\left[\sum_{i=1}^M cp_i \, u_1(0, S_t, r_t; t_i) + u_1(0, S_t, r_t; T)\right.$$
$$\left. +\eta\left(1 - u_1(0, S_t, r_t; T) - \int_0^T u_2(0, S_t, r_t; \tau_1) \, d\tau_1\right)\right]. \tag{6}$$

Moreover, applying the Feynman-Kac formula (see [16], for example) and by using the change of variable $y_t = r_t \exp(\kappa t)$, u_1 and u_2 are solutions of the Cauchy problem

$$\begin{cases} \overline{\mathcal{L}}\left[u(t,S,y)\right] = 0, & t < T_1, \ (S,y) \in (0,\infty) \times (-\infty,\infty), \\ u(T_1,S,y) = h(y), & (S,y) \in (0,\infty) \times (-\infty,\infty), \end{cases} \tag{7}$$

with $h(y) = 1$ for $u = u_1$ or $h(y) = \exp(-\kappa T_1)y$ for $u = u_2$, respectively. Moreover, the operator $\mathcal{L}$ is defined as follows

$$\begin{aligned} \overline{\mathcal{L}}[u] \;=\; & \partial_t u + \frac{1}{2}\sigma^2(t,S)S^2\,\partial_{SS}u + \rho\delta\sigma(t,S)\exp(\kappa t)S\,\partial_{Sy}u + \frac{1}{2}\delta^2\exp(2\kappa t)\,\partial_{yy}u \\ & + (\exp(-\kappa t)y + \lambda(t,S))\,S\,\partial_S u + \kappa\theta\exp(\kappa t)\,\partial_y u - (\exp(-\kappa t)y + \lambda(t,S))\,u. \end{aligned} \tag{8}$$

3. Existence of solutions

In this section, we aim to prove existence of a "local" solution to the Cauchy problem (7). For this purpose, let X be the logarithm of the pre-defaultable stock price, i.e. $X_t = \log S_t$, $t \in [0,T]$. Our model becomes

$$\begin{cases} \overline{S}_t = S_0 e^{X_t}\mathbb{1}_{\{\xi > t\}} & S_0 > 0 \\ dr_t = \kappa(\theta - r_t)dt + \delta W_t^1 \\ dX_t = (r_t - \frac{1}{2}\sigma(t,X_t)^2 + \lambda(t,X_t))dt + \sigma(t,X_t)dW_t^2 \\ dW_t^1 dW_t^2 = \rho dt, \end{cases} \tag{9}$$

with $\sigma(t,x) = a(t)e^{\beta x}$ and $\lambda(t,x) = b(t) + c\sigma(t,x)^2$. The correspondent infinitesimal generator is the operator $\mathcal{L}$ defined as

$$\begin{aligned} \mathcal{L} \;=\; & \partial_t + \frac{1}{2}\sigma^2(t,x)\,\partial_{xx} + \rho\delta\sigma(t,x)\,\partial_{xr} + \frac{1}{2}\delta^2\,\partial_{rr} \\ & + \left(r - \frac{1}{2}\sigma(t,x) + \lambda(t,x)\right)\partial_x + \kappa\,(\theta - r)\,\partial_r - (r + \lambda(t,x)) \end{aligned} \tag{10}$$

$$= \; \partial_t + \frac{1}{2}\langle \Sigma\nabla, \nabla\rangle + \langle \mu, \nabla\rangle + \gamma \tag{11}$$

where

$$\begin{aligned} \Sigma\,(t,x,r) \;&=\; \begin{pmatrix} \sigma^2\,(t,x) & \rho\delta\sigma\,(t,x) \\ \rho\delta\sigma\,(t,x) & \delta^2 \end{pmatrix}, \quad \mu\,(t,x,r) = \begin{pmatrix} r - \frac{1}{2}\sigma^2\,(t,x) + \lambda\,(t,x) \\ \kappa\,(\theta - r) \end{pmatrix}, \\ \gamma\,(t,x) \;&=\; -\,(r + \lambda\,(t,x)). \end{aligned} \tag{12}$$

Operator $\mathcal{L}$ is *only locally* uniformly parabolic in the sense that, for any ball

$$\mathcal{O}_R := \left\{ (x,r) \in \mathbb{R}^2 \mid |(x,r)| < R \right\};$$

the coefficients of $\mathcal{L}$ satisfy the following conditions:

(H1) The matrix $\Sigma(t,x,r)$ is positive definite, uniformly with respect to $(t,x,r) \in (0,T] \times \mathcal{O}_R$.

(H2) The coefficients Σ, μ, γ are bounded and Hölder-continuous on $(0,T] \times \mathcal{O}_R$.

Under these conditions we can resort on the recent results in [**?**], Theor. 2.6, or [**?**], Theor.1.5, about the existence of a *local density* for the process (X,r).

Theorem 3.1. *For any $R > 0$, the process (X,r) has a local transition density on $\mathcal{O}_R$, that is a non-negative measurable function $\Gamma = \Gamma(t,x,r;T,y,s)$ defined for any $0 < t < T$, $(x,r) \in \mathbb{R}^2$ and $(y,s) \in \mathcal{O}_R$ such that, for any continuous function $h = h(x,r)$ with compact support in $\mathcal{O}_R$, we have*

$$u(t,x,r) := E_{t,x,r}\left[h(X_T, r_T)\right] = \int_{\mathcal{O}_R} \Gamma(t,x,r;T,y,s)h(y,s)dy ds$$

and u satisfies

$$\begin{cases} \mathcal{L}u(t,x,r) = 0, & (t,x,r) \in [0,T) \times \mathcal{O}_R, \\ u(T,x,r) = h(x,r) & (x,r) \in \mathcal{O}_R. \end{cases} \tag{13}$$

Problem (13) can be used for numerical approximation purposes. Notice however that (13) does not posses a unique solution due to the lack of lateral boundary conditions. Nevertheless, numerical schemes can be implemented imposing artificial boundary conditions and the result which guarantees the validity of such approximations is the so-called *principle of not feeling the boundary*. A rigorous statement of this result can be found in [**?**], Appendix A, or [**?**], Lemma4.11.

4. Numerical methods

In order to obtain a numerical approach of the value of a non callable defaultable coupon bond we need to solve the Cauchy problem (7) for $u = u_1$ and $u = u_2$ with maturity $T_1 = t_i$ for $i = 1, ..., M$, that is each coupon payment date for both cases. Once these problems are solved, the value of the bond is given by expression (6) in which appears an integral term. That integral term will be approximated by means of the classical composite trapezoidal rule. For the numerical solution of the PDE problem, we propose a Crank-Nicolson characteristics time discretization scheme combined with a piecewise bi-quadratic Lagrange finite element method. This Lagrange-Galerkin method has been analyzed in [3, 4] for time and space discretization. More recently, it has been applied to the valuation of pension plans without and with early retirement in [5] and [6], respectively. Thus, in order to apply this numerical technique, first a localization procedure is used to cope with the initial formulation in an unbounded domain.

4.1. Localization procedure and formulation in a bounded domain

In this section we replace the unbounded domain by a bounded one. In order to determine the required boundary conditions for the associated PDE problem we follow [15] which is based on the theory proposed by Fichera in [12]. Let us introduce the following notation:

$$x_0 = t, \quad \tilde{x}_1 = S \quad \text{and} \quad \tilde{x}_2 = y. \tag{14}$$

For this purpose, let us consider both $\tilde{x}_1^\infty$ and $\tilde{x}_2^\infty$ to be large enough suitably chosen real numbers in order to the solution in the region of interest is not affected by the truncation of the domain. Let

$$\Omega^* = (0, x_0^\infty) \times \left(\frac{1}{\tilde{x}_1^\infty}, \tilde{x}_1^\infty \right) \times (-\tilde{x}_2^\infty, \tilde{x}_2^\infty)$$

with $x_0^\infty = T_1$. Additionally, we make the changes of variables $x_1 = \tilde{x}_1 - \frac{1}{\tilde{x}_1^\infty}$, $x_2 = \tilde{x}_2 + \tilde{x}_2^\infty$, $x_1^\infty = \tilde{x}_1^\infty - \frac{1}{\tilde{x}_1^\infty}$ and $x_2^\infty = 2\tilde{x}_2^\infty$, which leads to replace the bounded domain Ω^* by the following one:

$$\Omega = (0, x_0^\infty) \times (0, x_1^\infty) \times (0, x_2^\infty)$$

Then, let us denote the Lipschitz boundary by $\Gamma = \partial\Omega$ such that $\Gamma = \bigcup_{i=0}^2 (\Gamma_i^- \cup \Gamma_i^+)$, where

$$\Gamma_i^- = \{(x_0, x_1, x_2) \in \Gamma \mid x_i = 0\}, \quad \Gamma_i^+ = \{(x_0, x_1, x_2) \in \Gamma \mid x_i = x_i^\infty\}, \quad i = 0, 1, 2.$$

Then, the operator defined in (8) can be written in the form:

$$\mathcal{L}[u] = \sum_{i,j=0}^2 b_{ij} \frac{\partial^2 u}{\partial x_i x_j} + \sum_{j=0}^2 b_j \frac{\partial u}{\partial x_j} + b_0 u, \tag{15}$$

where the involved data are given by

$$\mathbf{B} = (b_{ij}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2}a^2(t)\left(x_1 + \frac{1}{\tilde{x}_1^\infty}\right)^{2\beta+2} & \frac{1}{2}\rho\delta a(t)\left(x_1 + \frac{1}{\tilde{x}_1^\infty}\right)^{\beta+1}\exp(\kappa t) \\ 0 & \frac{1}{2}\rho\delta a(t)\left(x_1 + \frac{1}{\tilde{x}_1^\infty}\right)^{\beta+1}\exp(\kappa t) & \frac{1}{2}\delta^2\exp(2\kappa t) \end{pmatrix}, \quad (16)$$

$$\mathbf{b} = (b_j) = \begin{pmatrix} 1 \\ \left(\exp(-\kappa t)(x_2 - \tilde{x}_2^\infty) + \lambda\left(t, x_1 + \frac{1}{\tilde{x}_1^\infty}\right)\right)\left(x_1 + \frac{1}{\tilde{x}_1^\infty}\right) \\ \kappa\theta\exp(\kappa t) \end{pmatrix}, \quad (17)$$

$$b_0 = -\left(\exp(-\kappa t)(x_2 - \tilde{x}_2^\infty) + \lambda\left(t, x_1 + \frac{1}{\tilde{x}_1^\infty}\right)\right). \quad (18)$$

Thus, following [15], in terms of the normal vector to the boundary pointing inward Ω, $\mathbf{m} = (m_0, m_1, m_2)$, we introduce the following subsets of Γ:

$$\Sigma^0 = \left\{x \in \Gamma / \sum_{i,j=0}^{2} b_{ij}m_i m_j = 0\right\}, \quad \Sigma^1 = \Gamma - \Sigma^0,$$

$$\Sigma^2 = \left\{x \in \Sigma^0 / \sum_{i=0}^{2}\left(b_i - \sum_{j=0}^{2}\frac{\partial b_{ij}}{\partial x_j}\right)m_i < 0\right\}.$$

As indicated in [15] the boundary conditions at $\Sigma^1 \bigcup \Sigma^2$ for the so-called first boundary value problem associated with (15) are required. Note that $\Sigma^1 = \{\Gamma_1^-, \Gamma_1^+, \Gamma_2^-, \Gamma_2^+\}$ and $\Sigma^2 = \{\Gamma_0^+\}$. Therefore, in addition to a final condition (see section 2.2), we need to impose boundary conditions on $\Gamma_1^-, \Gamma_1^+, \Gamma_2^-$ and Γ_2^+. Next, we will impose Dirichlet conditions on Γ_1^-, Γ_2^- and Γ_2^+, whereas on Γ_1^+ we will impose a homogeneous Neumann condition.

Taking into account the previous change of spatial variable and making the change of time variable $\tau = T_1 - t$, we write the equation (7) in divergence form in the bounded spatial domain $\Omega = (0, x_1^\infty) \times (0, x_2^\infty)$. Thus, the initial boundary value problem (IBVP) takes the following form:

Find $u : [0, T_1] \times \Omega \to \mathbb{R}$ such that

$$\partial_\tau u - Div(A\nabla u) + \mathbf{v}\cdot\nabla u + lu = f \quad \text{in } (0,T_1)\times\Omega, \quad (19)$$

$$u(0,.) = h(x_2 - \tilde{x}_2^\infty) \quad \text{in } \Omega, \quad (20)$$

$$u = \exp\left(-\int_{T_1-\tau}^{T_1}\left(\exp(-\kappa\tilde{u})(x_2 - \tilde{x}_2^\infty) + \lambda\left(\tilde{u}, x_1 + \frac{1}{\tilde{x}_1^\infty}\right)\right)d\tilde{u}\right)h(x_2 - \tilde{x}_2^\infty) \quad \text{on } (0,T_1)\times\Gamma_1^-, \quad (21)$$

$$\frac{\partial u}{\partial x_1} = 0 \quad \text{on } (0,T_1)\times\Gamma_1^+, \quad (22)$$

$$u = \exp\left(-\int_{T_1-\tau}^{T_1}\left(\exp(-\kappa\tilde{u})(x_2 - \tilde{x}_2^\infty) + \lambda\left(\tilde{u}, x_1 + \frac{1}{\tilde{x}_1^\infty}\right)\right)d\tilde{u}\right)h(x_2 - \tilde{x}_2^\infty) \quad \text{on } (0,T_1)\times\Gamma_2^-, \quad (23)$$

$$u = \exp\left(-\int_{T_1-\tau}^{T_1}\left(\exp(-\kappa\tilde{u})(x_2 - \tilde{x}_2^\infty) + \lambda\left(\tilde{u}, x_1 + \frac{1}{\tilde{x}_1^\infty}\right)\right)d\tilde{u}\right)h(x_2 - \tilde{x}_2^\infty) \quad \text{on } (0,T_1)\times\Gamma_2^+. \quad (24)$$

For problem (19)-(24), the diffusion matrix $\mathbf{A}$, the velocity field $\mathbf{v}$, the reaction function l and the second

member f are defined as follows:

$$\mathbf{A} = \begin{pmatrix} \frac{1}{2}a^2(T_1-\tau)\left(x_1+\frac{1}{\tilde{x}_1^\infty}\right)^{2\beta+2} & \frac{1}{2}\rho\delta a(T_1-\tau)\left(x_1+\frac{1}{\tilde{x}_1^\infty}\right)^{\beta+1}\exp(\kappa(T_1-\tau)) \\ \frac{1}{2}\rho\delta a(T_1-\tau)\left(x_1+\frac{1}{\tilde{x}_1^\infty}\right)^{\beta+1}\exp(\kappa(T_1-\tau)) & \frac{1}{2}\delta^2\exp(2\kappa(T_1-\tau)) \end{pmatrix},$$

$$\mathbf{v} = \begin{pmatrix} \frac{1}{2}a^2(T_1-\tau)(2\beta)\left(x_1+\frac{1}{\tilde{x}_1^\infty}\right)^{2\beta+1} - \left(\exp(-\kappa(T_1-\tau))(x_2-\tilde{x}_2^\infty)+\lambda\left(T_1-\tau,x_1+\frac{1}{\tilde{x}_1^\infty}\right)\right)\left(x_1+\frac{1}{\tilde{x}_1^\infty}\right) \\ \frac{1}{2}\rho\delta a(T_1-\tau)(\beta)\left(x_1+\frac{1}{\tilde{x}_1^\infty}\right)^{\beta-1}\exp(\kappa(T_1-\tau))-\kappa\theta\exp(\kappa(T_1-\tau)) \end{pmatrix},$$

$$l = \exp(-\kappa(T_1-\tau))(x_2-\tilde{x}_2^\infty)+\lambda\left(T_1-\tau,x_1+\frac{1}{\tilde{x}_1^\infty}\right),$$

$$f = 0.$$

4.2. Time discretization

The method of characteristics is based on a finite differences scheme for the discretization of the material derivative, i.e., the time derivative along the characteristic lines of the convective part of the equation (19). The material derivative operator is given by

$$\frac{D}{D\tau} = \partial_\tau + \mathbf{v}\cdot\nabla.$$

For a brief description of the method, we first define the characteristics curve through $\mathbf{x}=(x_1,x_2)$ at time $\bar{\tau}$, $X(\mathbf{x},\bar{\tau};s)$, which satisfies:

$$\frac{\partial}{\partial s}X(\mathbf{x},\bar{\tau};s) = \mathbf{v}(X(\mathbf{x},\bar{\tau};s)), \quad X(\mathbf{x},\bar{\tau};\bar{\tau}) = \mathbf{x}. \tag{25}$$

In order to discretize in time the material derivative in the Cauchy problem (19), let us consider a number of time steps N, the time step $\Delta\tau=T/N$ and the time mesh points $\tau^n=n\Delta\tau$, $n=0,\frac{1}{2},1,\frac{3}{2},\ldots,N$.

The material derivative approximation by the characteristics method for both problems is given by:

$$\frac{Du}{D\tau} = \frac{u^{n+1}-u^n\circ X^n}{\Delta\tau},$$

where $u=u_1,u_2$ and $X^n(\mathbf{x})=X(\mathbf{x},\tau^{n+1};\tau^n)$. In this case, the solution of (25) are not computed analytically. Instead, we consider numerical ODE solvers to approximate the characteristics curves (see [3], for example). More precisely, in this work we employ the second order Runge-Kutta method.

Next, we consider a Crank-Nicolson scheme around $\left(X(\mathbf{x},\tau^{n+1};\tau),\tau\right)$ for $\tau=\tau^{n+\frac{1}{2}}$. So, the time discretized equation for $u=u_1,u_2$ can be written as follows:

Find u^{n+1} such that:

$$\frac{u^{n+1}(\mathbf{x})-u^n(X^n(\mathbf{x}))}{\Delta\tau} - \frac{1}{2}Div(A\nabla u^{n+1})(\mathbf{x}) - \frac{1}{2}Div(A\nabla u^n)(X^n(\mathbf{x}))$$
$$+\frac{1}{2}(l\,u^{n+1})(\mathbf{x})+\frac{1}{2}(l\,u^n)(X^n(\mathbf{x})) = 0. \tag{26}$$

In order to obtain the variational formulation of the semi-discretized problem, we multiply (26) by a suitable

test function, integrate in Ω, use the classical Green formula and the following one [14]:

$$\int_\Omega Div(\mathbf{A}\nabla u^n)(X^n(\mathbf{x}))\psi(\mathbf{x})d\mathbf{x} \;=\; \int_\Gamma (\nabla X^n)^{-T}(\mathbf{x})\mathbf{n}(x)\cdot(\mathbf{A}\nabla u^n)(X^n(\mathbf{x}))\psi(\mathbf{x})dA_\mathbf{x}$$
$$-\int_\Omega (\nabla X^n)^{-1}(\mathbf{x})(\mathbf{A}\nabla u^n)(X^n(\mathbf{x}))\cdot\nabla\psi(\mathbf{x})d\mathbf{x}$$
$$-\int_\Omega Div((\nabla X^n)^{-T}(\mathbf{x}))\cdot(\mathbf{A}\nabla u^n)(X^n(\mathbf{x}))\psi(\mathbf{x})d\mathbf{x}.$$

(27)

Note that, due to the characteristics curves can not be obtained analytically, the terms $(\nabla X^n)^{-1}(\mathbf{x})$ and $Div((\nabla X^n)^{-T}(\mathbf{x}))$ in (27) are replaced by the following approximations (see [3] for more detail):

$$(\nabla X^n)^{-1}(\mathbf{x}) = \mathbf{I}(\mathbf{x}) + \Delta\tau\mathbf{L}^n(X^n(\mathbf{x})) + O(\Delta\tau^2),$$

$$Div((\nabla X^n)^{-T}(\mathbf{x})) = \Delta\tau\,\nabla\,Div\,(\mathbf{v}^n(X^n(\mathbf{x}))) + O(\Delta\tau^2),$$

where $\mathbf{L} = \nabla\mathbf{v}$.

After the previous steps, we can write a variational formulation for the time discretized problem as follows:

Find $u^{n+1} \in H^1(\Omega)$ for all $\psi \in H^1(\Omega)$ such that $\psi = 0$ on Γ_1^-, Γ_2^- and Γ_2^+ :

$$\frac{1}{\Delta\tau}\int_\Omega u^{n+1}(\mathbf{x})\psi(\mathbf{x})d\mathbf{x} + \frac{1}{2}\int_\Omega(\mathbf{A}\nabla u^{n+1})(\mathbf{x})\cdot\nabla\psi(\mathbf{x})d\mathbf{x} + \frac{1}{2}\int_\Omega(lu^{n+1})(\mathbf{x})\psi(\mathbf{x})d\mathbf{x}$$
$$= \frac{1}{\Delta\tau}\int_\Omega u^n(X^n(\mathbf{x}))\psi(\mathbf{x})d\mathbf{x} - \frac{1}{2}\int_\Omega(\mathbf{A}\nabla u^n)(X^n(\mathbf{x}))\cdot\nabla\psi(\mathbf{x})d\mathbf{x}$$
$$-\frac{\Delta\tau}{2}\int_\Omega\mathbf{L}^n(X^n(\mathbf{x}))(\mathbf{A}\nabla u^n)(X^n(\mathbf{x}))\cdot\nabla\psi(\mathbf{x})d\mathbf{x} - \frac{1}{2}\int_\Omega(lu^n)(X^n(\mathbf{x}))\psi(\mathbf{x})d\mathbf{x}$$
$$-\frac{\Delta\tau}{2}\int_\Omega\nabla\,Div\,(\mathbf{v}^n(X^n(\mathbf{x})))\cdot(\mathbf{A}\nabla u^n)(X^n(\mathbf{x}))\psi(\mathbf{x})d\mathbf{x}$$
$$+\frac{1}{2}\int_\Gamma(\mathbf{I}(\mathbf{x}) + \Delta\tau\mathbf{L}^n(X^n(\mathbf{x})))^T\mathbf{n}(x)\cdot(\mathbf{A}\nabla u^n)(X^n(\mathbf{x}))\psi(\mathbf{x})dA_\mathbf{x} + \frac{1}{2}\int_{\Gamma_1^+}a_{12}(\mathbf{x})\frac{\partial u}{\partial x_2}(\mathbf{x})\psi(\mathbf{x})dA_\mathbf{x},$$

(28)

where a_{12} is the corresponding coefficient of the diffusion matrix $\mathbf{A}$.

4.3. Finite elements discretization

For the spatial discretization we consider $\{\tau_h\}$, a quadrangular mesh of the domain Ω. Let $(T_2, \mathcal{Q}_2, \Sigma_{T_2})$ be a family of piecewise quadratic Lagrangian finite elements, where $\mathcal{Q}_2$ denotes the space of polynomials defined in $T_2 \in \tau_h$ with degree less or equal than two in each spatial variable and Σ_{T_2} the subset of nodes of the element T_2. More precisely, let us define the finite elements space u_h by

$$u_h = \{\phi_h \in \mathcal{C}^0(\Omega) : \phi_{h_{T_2}} \in \mathcal{Q}_2, \quad \forall T_2 \in \tau_h\},$$

(29)

where $\mathcal{C}^0(\Omega)$ is the space of piecewise continuous functions on Ω.

4.4. Composite trapezoidal rule

In order to obtain the value of the bond at origination by means of expression (6) the computation of an integral term is required. The approximation of this integral is carried out by using a suitable numerical integration procedure. More precisely, we employ the classical composite trapezoidal rule with $M + 1$ points, where M is the number of coupons, in the following way:

$$\int_0^T u_2(0, S_t, r_t; \tau_1)d\tau_1 \approx \frac{h}{2}\left[u_2(0, S_t, r_t; 0) + 2\sum_{j=1}^{M-1} u_2(0, S_t, r_t; k_j) + u_2(0, S_t, r_t; T)\right]$$

(30)

where $h = \frac{T}{M}$, $k_j = jh$ for $j = 1, ..., M - 1$ and $u_2(0, S_t, r_t; 0) = r_0$.

5. Numerical results

In order to show the good performance of the numerical methods explained in Section 4, we present some numerical results. In the following examples, the value of some of the parameters involved in the underlying factors are taken from the literature, in particular from [11] where the authors calibrate the model parameters to market data. More precisely, first the interest rate model is calibrated to zero-coupon bonds (ZCB) and next the model is calibrated to CDS spreads whose price is obtained by means of an asymptotic expansion method.

In both examples, the number of elements and nodes of the finite element meshes employed in the numerical solution of the problems are shown in Table 1.

5.1. Example 1

First, we consider the simple case of the valuation of default-free zero-coupon bonds with different maturities. In this setting, the valuation problem is reduced to a one-factor model. The purpose of this example is to compare the value of the bonds we obtain with the market zero-coupon curve. In order to obtain the value of the bonds, we solve the IBVP (19) with initial condition $h = 1$ and only taking into account as underlying factor the interest rate. The value of the parameters involved in the interest rate model are the ones collected in Table 2. The values of the zero-coupon curve and the approximated ones obtained by solving the here proposed model are presented in Table 3. For the numerical solution we consider Mesh 64 and the time step $\Delta\tau = \frac{1}{360}$ (one day).

	Number of elements	Number of nodes
Mesh 4	16	81
Mesh 8	64	289
Mesh 16	256	1089
Mesh 32	1024	4225
Mesh 64	4096	16641

Table 1: Different finite element meshes (number of elements and nodes).

Parameters of the defaultable stock price model
$a_1 = 0.0337851$
$a_2 = 0.0523625$
$b_1 = 0.0026639$
$b_2 = 0.0027968$
$c = 0.0435673$
$\beta = -0.268496$
Parameters of the interest rate model
$\kappa = 0.04520533766268042$
$\delta = 0.02146900332086033$
$\theta = 0.10334921942765922$
Correlation coefficient
$\rho = 0.0$
Initial conditions
$S_0 = 1.0$
$r_0 = -0.009159871729892612$

Table 2: Parameters of the model for the UBS bond.

Maturity (years)	Market ZCB	Model ZCB
1	1.00229	1.006751
2	1.00372	1.009062
3	1.00333	1.007495
4	1.00099	1.002601
5	0.995825	0.994902
6	0.987805	0.984889
7	0.976833	0.973024
8	0.963223	0.959738
9	0.947687	0.945429
10	0.932845	0.930463

Table 3: Market and model values of the ZCB.

5.2. Example 2

Next, we consider the valuation of two real defaultable bonds traded in the market and issued by different firms. For this purpose, as in [11], we assume that the coefficients $a(t)$ and $b(t)$ in (2) are linearly dependent on time and their expresions are given by $a(t) = a_1 t + a_2$, $b(t) = b_1 t + b_2$, where a_1, a_2, b_1 and b_2 are constants.

On one hand, we take into account the pricing of a bond from UBS with maturity 5 years and a face value of 100. The bond pays annually coupon rates of 1.25 basis points and the recovery rate at the event of default is 40%. The model parameter values for this example are collected in Table 2. In this case the correlation coefficient ρ is assumed to be zero. Next, in Table 4 we present the value of the bond for different meshes and time steps. In this case, we can appreciate that the price of the bond converges to 102.62.

On the other hand, we present a correlated case. More precisely, we address the valuation of a bond from JP Morgan with maturity 5 years and a face value of 100. The bond pays annually coupon rates of 3.25 basis points and the recovery rate at the event of default is again 40%. For this second example the parameter values of the model are the ones which appear in Table 5. As we have just pointed out the correlation coefficient ρ is different from zero. Finally, the value of this bond is shown in Table 6. In this case, the value of the bond converges to 103.57.

In both cases, in order to obtain the value of the bond for we need to solve for each coupon payment date the IBVP (19) with maturity those dates and with initial condition $h = 1$ or $h = \exp(-\kappa T)(x_2 - \tilde{x}_2^\infty)$. More precisely, in both examples the maturity of the bond is 5 years and the frequency of coupon payments is annually, thus to obtain the value of the bond we need to solve the problem (19) 10 times, i.e. 5 times with one initial condition to obtain the value of u_1 and 5 times with the other initial condition to obtain the value of u_2.

Next, in Figures 1 and 2 we show the mesh value of the UBS bond and the JP Morgan bond, respectively. Both figures are obtained with the finer mesh and with time step $\Delta\tau = \frac{1}{360}$ (one day).

Time steps per year	Mesh 4	Mesh 8	Mesh 16	Mesh 32	Mesh 64
90	102.499496	102.603837	102.616859	102.619028	102.618959
180	102.499599	102.605953	102.617976	102.619681	102.619602
360	102.500454	102.606351	102.618421	102.620069	102.619925
720	102.500663	102.606518	102.618587	102.620248	102.620094

Table 4: Value of the UBS bond for different meshes and time steps.

Parameters of the defaultable stock price model
$a_1 = 0.0312763$
$a_2 = 0.0356952$
$b_1 = 0.00038362$
$b_2 = 0.00172115$
$c = 0.346622$
$\beta = -0.223027$
Parameters of the interest rate model
$\kappa = 0.14485883018483803$
$\delta = 0.01330207057173363$
$\theta = 0.03467342840511061$
Correlation coefficient
$\rho = 0.497108$
Initial conditions
$S_0 = 1.0$
$r_0 = 0.01469383913023823$

Table 5: Parameters of the model for the JP Morgan bond.

Time steps per year	Mesh 4	Mesh 8	Mesh 16	Mesh 32	Mesh 64
90	103.725041	103.596891	103.572191	103.570225	103.570524
180	103.841153	103.605155	103.575270	103.572747	103.573144
360	103.794567	103.602389	103.576483	103.574147	103.574509
720	103.794298	103.602461	103.577110	103.574772	103.575161

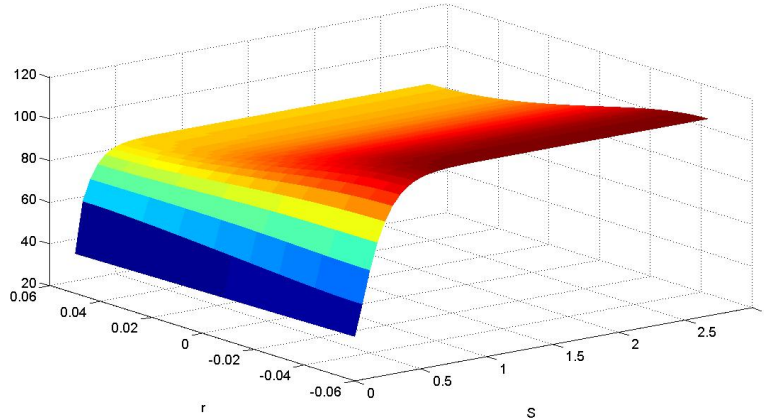Table 6: Value of the JP Morgan bond for different meshes and time steps.

Figure 1: Value of the UBS bond

6. Conclusions

In this paper we have considered the valuation of a non callable defaultable coupon bond where the underlying stochastic factors are the interest rate and the defaultable stock price. The pricing problem is posed as a sequence of IBVPs. More precisely, two PDE problems with different initial conditions with maturity each coupon payment date need to be solved. Once the numerical solution of these problems is carried out, the value of the bond is computed by means of an expression which also involves the computation of an integral term.

In order to obtain a numerical solution of the PDE problems, we have proposed appropriate numerical methods
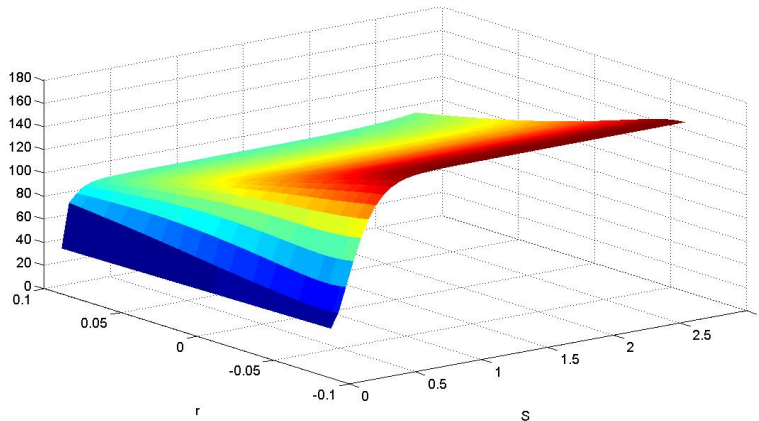
Figure 2: Value of the JP Morgan bond

based on Lagrange-Galerkin formulations. More precisely, we combine a Crank-Nicolson semi-Lagrangian scheme for time discretization with biquadratic Lagrange finite elements for space discretization. Moreover, the integral term which is involved in the computation of the bond value is approximated by means of the classical composite trapezoidal rule. Finally, we show some numerical results in order to illustrate the behaviour of the proposed methods.

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