Additive Approximation Schemes for Load Balancing problems

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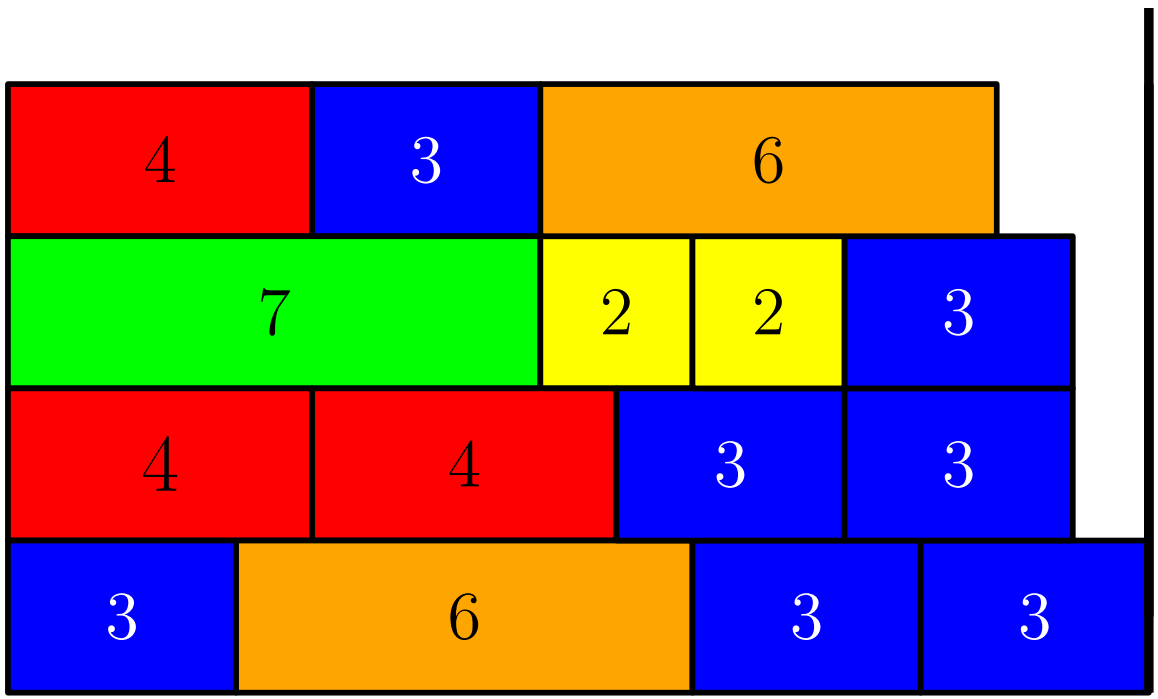
- **Goal:**
  1. *Makespan minimization:* $\min \max_i \{C_i\}$
  2. *Santa Claus problem:* $\max \min_i \{C_i\}$
  3. *Envy-minimizing Santa Claus problem:* $\min \max_i \{C_i\} - \min_i \{C_i\}$

Makespan minimization

Objective:

$$\min \max_i \{C_i\}$$

$$C_{\max} = 15$$

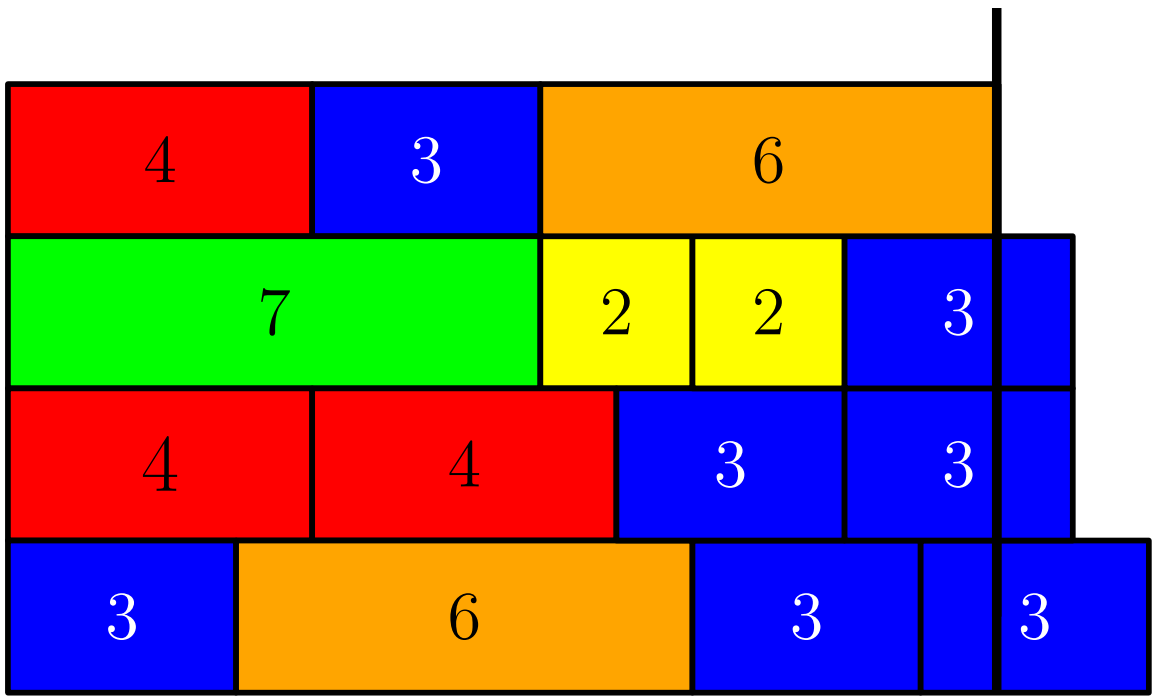


Santa Claus

Objective:

$$\max \min_i \{C_i\}$$

$$C_{\min} = 13$$

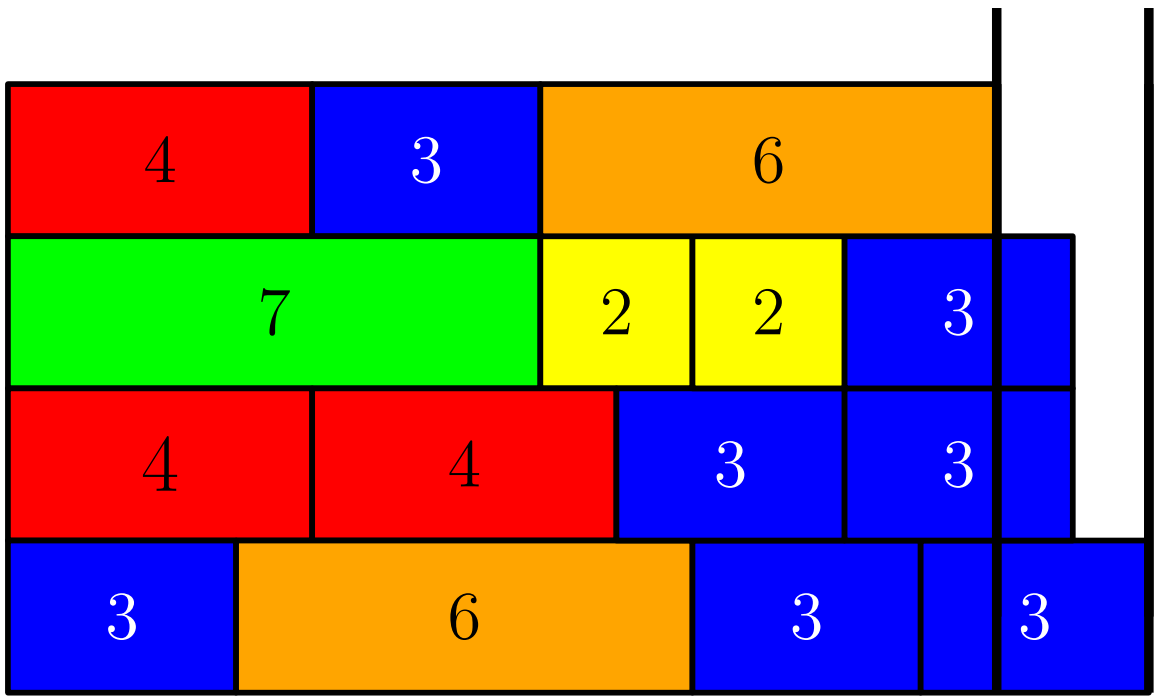


Envy-minimizing Santa Claus

Objective:

$$\min \max_i \{C_i\} - \min_i \{C_i\}$$

$$C_{\max} - C_{\min} = 2$$



Related work

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- **Approximation results:**
  - *Makespan minimization and Santa Claus:* (Efficient) polynomial time approximation schemes exist [1, 2, 4, 5, 6, 7]
  - *Envy-minimizing Santa Claus:* No constant factor approximation possible, unless $P = NP$

Our result

- **Additive approximation schemes:** Given $\epsilon > 0$, can we find solutions with an additive performance guarantee depending on the maximum processing time and ϵ, i.e.

$$|\mathrm{ALG} - \mathrm{OPT}| \leq \epsilon p_{\max}$$

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Theorem (Additive polynomial time approximation scheme)

For any $\epsilon > 0$, we can find a solution for the makespan minimization, Santa Claus or envy-minimizing Santa Claus problem with a value ALG satisfying

$$|\mathrm{ALG} - \mathrm{OPT}| \leq \epsilon p_{\max}$$

in running time $m^{O\left(2^{1/\epsilon}\right)} \cdot n^{O\left(1/\epsilon \cdot 2^{1/\epsilon}\right)}$.

Additive approximation scheme for makespan minimization

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2. Derive **structural properties** of optimal solutions to the slot-MILP and use these
3. Using **local search** to **round** the slot-MILP solution in order to obtain a schedule/assignment

The slot-MILP relaxation

- **Job types:** Partition $\mathcal{J}$ into $\mathcal{J}_1, \ldots, \mathcal{J}_{1/\epsilon}$, where

$$\mathcal{J}_k = \{j \in J | p_j \in ((k-1)\epsilon \cdot p_{\max}, k\epsilon \cdot p_{\max}]\}.$$

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- **Integer slot variables:** Number of slots for jobs of type k on machine i.
- **Fractional assignment variables:** Assignment of job j to machine i.

$$\begin{aligned}
&\min u && \\
&\sum_{i \in \mathcal{M}} x_{i,j} = 1 && \forall j \in \mathcal{J} \\
&\sum_{j \in \mathcal{J}} p_j x_{i,j} \leq u && \forall i \in \mathcal{M} \\
&\sum_{j \in \mathcal{J}_k} x_{i,j} = y_{i,k} && \forall i \in \mathcal{M}, \forall k \in \{1, \ldots, 1/\epsilon\} \\
&x_{i,j} \geq 0 && \forall j \in \mathcal{J}, i \in \mathcal{M} \\
&y_{i,k} \in \mathbb{N}_0 && \forall i \in \mathcal{M}, k \in \{1, \ldots, 1/\epsilon\}
\end{aligned}$$

Solving the slot-MILP: Structural property

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- **Negative:** Slot-MILP contains $m \cdot \frac{1}{\epsilon}$ many integer variables so we can not apply results for (M)ILPs of fixed dimension
- **Positive:** We can reduce the number of slot vectors $(y_i = (y_{i,1}, \ldots, y_{i,\frac{1}{\epsilon}}))$ to be considered

Solving the slot-MILP: Structural property

Lemma

There is an optimal solution (x, y) to the slot-MILP such that for all $i, i' \in \mathcal{M}$ with $y_i \equiv y_{i'} \bmod 2$ it follows that $y_i = y_{i'}$.

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Proof.

- Let (x, y) be an optimal solution in which for $i_1, i_2 \in \mathcal{M}$ and $y_{i_1} \equiv y_{i_2} \bmod 2$ it follows that $y_{i_1} \neq y_{i_2}$. Suppose that (x, y) minimizes potential function.

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- New solution (x', y'): take the average of machines i_1 and i_2 and leave all other machines unchanged.

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  - Potential function is decreased

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Solving the slot-MILP: Exact solution

Lemma

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- Total guesses: $m^{O\left(2^{1/\epsilon}\right)} \cdot n^{O\left(1/\epsilon \cdot 2^{1/\epsilon}\right)}$
- For each guess: Find optimal assignment variables and continue with the guess yielding the smallest u

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Local Search!

Rounding via Local Search - Fixing overloaded machines

1 **Initial solution:** Assign $y_{i,k}$ arbitrary jobs of type k to machine i. This may lead to overloaded machines:

$$> u + \epsilon p_{\max}$$

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      - Swap j and j' between i and i'.
      - If for any such pair the load on machine i' is more than u, let $\mathcal{M}_2$ be the set of these machines.
   2. Repeat the procedure until for $\mathcal{M}_1, \ldots, \mathcal{M}_\ell$ we can find a pair of jobs to swap between a machine in $\mathcal{M}_\ell$ and a not labelled machine yet. After that restart with $\mathcal{M}_1$.

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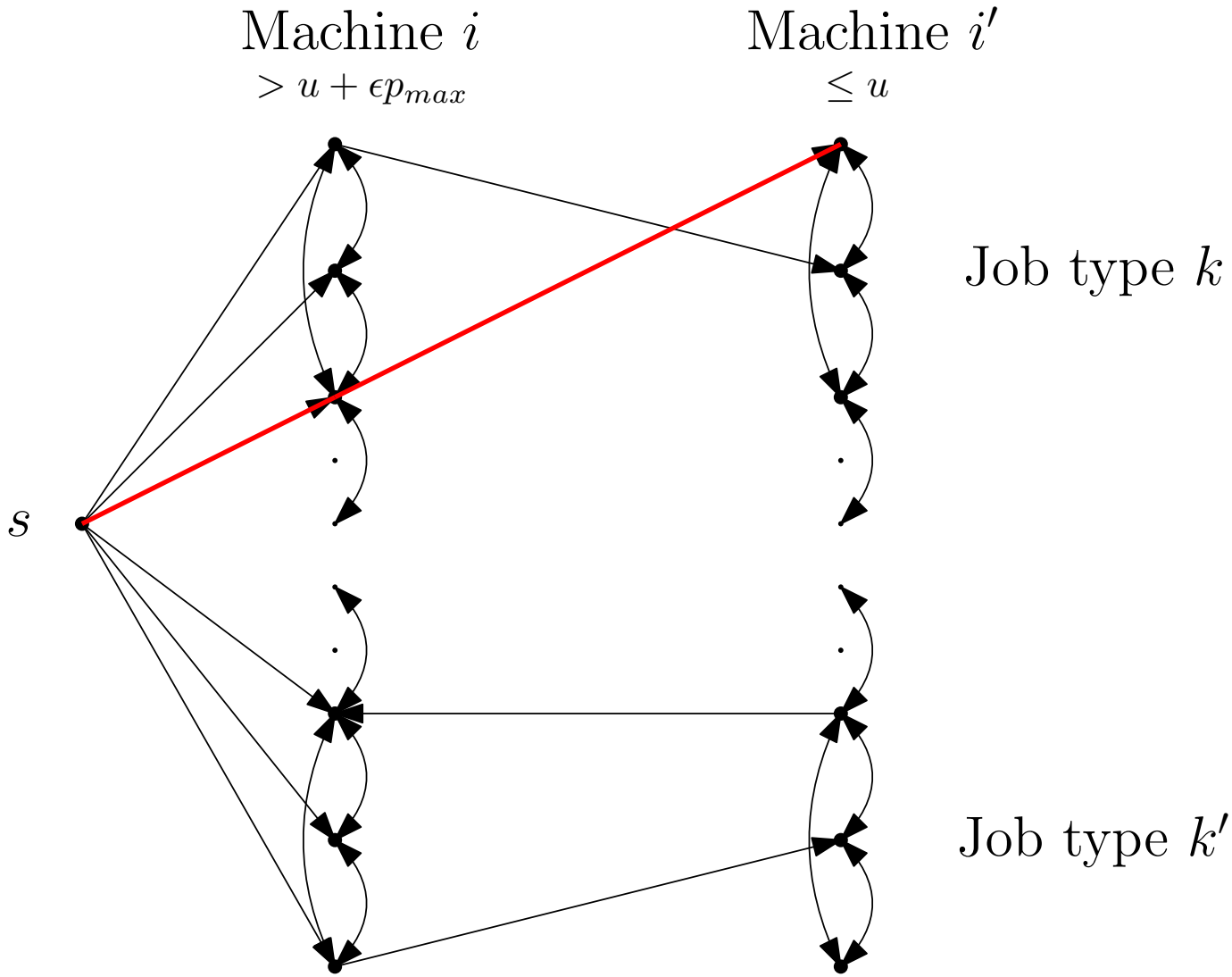
Iteration: Find a path from s to a slot on a machine with load at most u and swap the jobs associated to the final slot and the slot before that

Breadth First Search - Visualization



Rounding the slot-MILP

Lemma

Given an optimal solution (x, y) to the slot-MILP, in time $n^{O(1)}$ we can compute an integral solution (x', y) to slot-MILP such that for each machine $i \in \mathcal{M}$ we have:

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Proof.

1. **Existence of swappable jobs:** In each iteration a pair of swappable jobs can be found in time $O(n^2)$.
2. **Number of swaps:** After at most $O(n^3)$ iterations every machine satisfies the load upper bound.

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- **Improvement:** Using an alternative structural property we can improve the running time

Theorem

For any $\epsilon > 0$, we can find a solution for the makespan minimization, Santa Claus or envy-minimizing Santa Claus problem with a value ALG satisfying

$$|\mathrm{ALG} - \mathrm{OPT}| \leq \epsilon p_{\max}$$

in running time $m^2\left(\frac{n}{\epsilon^2}\right)^{O(1/\epsilon)}$.

Future Research

- **Improvement** of running time to obtain additive efficient polynomial time approximation schemes

Future Research

- **Improvement** of running time to obtain additive efficient polynomial time approximation schemes
- **Extension** of additive approximation schemes to other problems, e.g. sum of machine tardiness

Thank you

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