# XVA in a multi-currency setting with stochastic foreign exchange rates

Roberta Simonella[1,2]

Joint work with:

Carlos Vázquez[1,2]

[1] Department of Mathematics, University of A Coruña, 15071 A Coruña, Spain
[2] CITIC, 15071 A Coruña, Spain

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# Framework

- Trade between a non-defaultable hedger (H) and a defaultable investor (I)
- Multi-ccy framework: domestic ccy $D$, foreign ccys $C_0, \dots, C_N$
- Dynamics under the risk neutral probability measure of the domestic market:

$$\text{Underlying assets:} \quad dS_t^i = (r^i - q^i - \rho^{S^i,X^i}\sigma^{S^i}\sigma^{X^i})S_t^i\,dt + \sigma^{S^i}S_t^i\,dW_t^{S^i}, \quad i = 1,\dots,N$$

$$\text{FX rates:} \quad dX_t^{D,C_j} = (r^D - r^j)X_t^{D,C_j}\,dt + \sigma^{X^j}X_t^{D,C_j}\,dW^{X^j}, \quad j = 0,\dots,N$$

$$\text{I's credit spread:} \quad dh_t = (\mu^h - M^h\sigma^h)\,dt + \sigma^h\,dW_t^h$$

- I's intensity of default: $\lambda = \frac{h}{1-R}$, where $R$ is the I's recovery rate

$$\mu^h - M^h\sigma^h = -\kappa\lambda \quad \Rightarrow \quad dh_t = \frac{-\kappa}{1-R}h_t\,dt + \sigma^h\,dW_t^h$$

# Framework

- $S_t = (S_t^1, \dots, S_t^N)$, $X_t = (X_t^{D,C_0}, \dots, X_t^{D,C_N})$, $\bar{X}_t = (X_t^{D,C_1}, \dots, X_t^{D,C_N})$
- I's default state at time $t$:

$$J_t = \begin{cases} 1 & \text{in case of default before or at time } t \\ 0 & \text{otherwise} \end{cases}$$

- Derivative value in ccy $D$ at time $t$:
  - Risky: $V_t = V(t, S_t, X_t, h_t, J_t)$
  - Risk-free: $W_t = W(t, S_t, \bar{X}_t)$
- Mark-to-market derivative price: $M(t, S_t, X_t, h_t)$
- In case that I defaults:

$$V(t, S_t, X_t, h_t, 1) = RM^+(t, S_t, X_t, h_t) + M^-(t, S_t, X_t, h_t)$$

- Variation of $V$ at default:

$$\Delta V = RM^+ + M^- - V$$

# Building the replicating portfolio

## Risk factors

Self-financing portfolio $\Pi$ that hedges all the risk factors:

- market risk due to changes in $S^1$, $S^2, \ldots, S^N$
  $\Rightarrow$ fully collateralized derivatives on the same underlying assets,
  net present value $H^i$ in ccy $C_i$, $H^{i,D} = H^i X^{D/C_i}$ in ccy $D$
- FX risk due to changes in $X^{D,C_0}, \ldots, X^{D,C_N}$
  $\Rightarrow$ FX derivatives,
  net present value $E^j$ in ccy $D$
- $\mathrm{I}$'s spread risk due to changes in $h$ and $\mathrm{I}$'s default risk
  $\Rightarrow$ two credit default swaps with different maturities written on $\mathrm{I}$:
  - short term (overnight) credit default swap $CDS(t, t + dt)$
  - long term credit default swap $CDS(t, T)$

## Collateral account

Collateral account $C^{C_0}$ composed of a portfolio of bonds $R^{C_0}$ and cash $M^{C_0}$ (ccy $C_0$)

## Self-financing condition of a replicating strategy

The hedger matches the spread duration of the uncollateralized part of the derivative by trading on short term bonds: $\Omega_t B(t, t + dt) = V_t - C_t^{C_0} X_t^{D/C_0}$

# Building the replicating portfolio

## Replicating portfolio

$$\Pi_t = \sum_{i=1}^{N} \alpha_t^i H_t^i + \sum_{j=0}^{N} \eta_t^j E_t^j + \gamma_t CDS(t,T) + \epsilon_t CDS(t,t+dt) + \Omega_t B(t,t+dt) + \beta_t$$

## Bank account composition

$$\beta = -\sum_{i=1}^{N} \alpha^i H^{i,D} - \sum_{j=0}^{N} \eta^j E^j - \gamma CDS(t,T) + C^{C_0} X^{D/C_0}$$

Variation in the time interval $[t, t+dt]$:

$$d\beta_t = -\left[\sum_{i=1}^{N} \alpha_t^i (c^D + b^{D,C_j}) H_t^{i,D} + \sum_{j=0}^{N} \eta_t^j c^D E_t^j + \gamma_t c^D CDS(t,T)\right] dt + \left[\left(r^R + b^{D,C_0}\right) R_t^{C_0} + \left(c^D + b^{D,C_0}\right) M_t^{C_0}\right] X^{D/C_0} dt\,,$$

where $r^R$ is the instantaneous repo rate associated to the bond $R^{C_0}$, $b^{D,C_0}$ is the cross-ccy basis, and $c^D$ is the OIS rate in ccy $D$.

# Building the pricing PDE

**No arbitrage + self-financing condition**

$$V(t, S_t, h_t, J_t) = \Pi_t \quad \Rightarrow \quad dV_t = d\Pi_t$$

$$\begin{aligned}
&\frac{\partial V}{\partial t}dt + \sum_{i=1}^{N}\frac{\partial V}{\partial S^i}dS_t^i + \sum_{j=0}^{N}\frac{\partial V}{\partial X^j}dX_t^j + \frac{\partial V}{\partial h}dh_t + \Delta V dJ_t \\
&+ \left[\frac{1}{2}\sum_{i,k=1}^{N}\rho^{S^iS^k}\sigma^{S^i}\sigma^{S^k}S_t^iS_t^k\frac{\partial^2 V}{\partial S^i S^k} + \frac{1}{2}\sum_{j,l=0}^{N}\rho^{X^jX^l}\sigma^{X^j}\sigma^{X^l}X_t^jX_t^l\frac{\partial^2 V}{\partial X^j X^l} + \frac{1}{2}(\sigma^h)^2\frac{\partial^2 V}{\partial h^2}\right. \\
&\left. + \sum_{i=1}^{N}\sum_{j=0}^{N}\rho^{S^iX^j}\sigma^{S^i}\sigma^{X^j}S_t^iX_t^j\frac{\partial^2 V}{\partial S^i\partial X^j} + \sum_{i=1}^{N}\rho^{S^ih}\sigma^{S^i}\sigma^hS_t^i\frac{\partial^2 V}{\partial S^i\partial h} + \sum_{j=0}^{N}\rho^{X^jh}\sigma^{X^j}\sigma^hX_t^j\frac{\partial^2 V}{\partial X^j\partial h}\right]dt \\
&= \sum_{i=1}^{N}\alpha_t^i dH_t^{i,D} + \sum_{j=0}^{N}\eta_t^j dE_t^j + \gamma_t dCDS(t,T) + \epsilon_t dCDS(t, t+dt) + \Omega_t dB(t, t+dt) + d\beta_t
\end{aligned}$$

# Building the pricing PDE

$$dH_t^i = \left[ \frac{\partial H^{i,D}}{\partial t} + \frac{(S^i \sigma^{S^i})^2}{2} \frac{\partial^2 H^{i,D}}{\partial (S^i)^2} + \frac{(X^i \sigma^{X^i})^2}{2} \frac{\partial^2 H^{i,D}}{\partial (X^i)^2} + \rho^{S^i X^i} \sigma^{S^i} \sigma^{X^i} S^i X^i \frac{\partial^2 H^{i,D}}{\partial S^i \partial X^i} \right] + \frac{\partial H^i}{\partial S^i} dS_t^i + \frac{\partial H^i}{\partial X^i} dX_t^i$$

$$dE_t^j = \left[ \frac{\partial E^j}{\partial t} + \frac{(\sigma^{X^j} X_t^j)^2}{2} \frac{\partial^2 E^j}{\partial (X^j)^2} \right] dt + \frac{\partial E^j}{\partial X^j} dX_t^j$$

$$dCDS(t,T) = \left[ \frac{\partial CDS(t,T)}{\partial t} + \frac{(\sigma^h)^2}{2} \frac{\partial^2 CDS(t,T)}{\partial h^2} \right] dt + \frac{\partial CDS(t,T)}{\partial h} dh_t + \Delta CDS(t,T) dJ_t$$

$$dCDS(t, t+dt) = h_t\, dt - (1-R)\, dJ_t$$

$dB(t, t+dt) = f^{H,D} B(t, t+dt)\, dt$, $\quad f^{H,D}$ is the hedger's domestic funding rate

$$\alpha_t^i = \frac{\partial V}{\partial S^i} \Big/ \frac{\partial H^{i,D}}{\partial S^i} \qquad \eta_t^0 = \frac{\partial V}{\partial X^0} \Big/ \frac{\partial E^0}{\partial X^0}, \qquad \eta_t^i = \left( \frac{\partial V}{\partial X^i} - \alpha_t^i \frac{\partial H^{i,D}}{\partial X^i} \right) \Big/ \frac{\partial E^i}{\partial X^i}$$

$$\gamma_t = \frac{\partial V}{\partial h} \Big/ \frac{\partial CDS(t,T)}{\partial h} \qquad \epsilon_t = (\gamma_t \Delta CDS(t,T) - \Delta V) \big/ (1-R)$$

# Pricing PDE

**Pricing PDE for a generic mark-to-market value ($\Delta V = RM^+ + M^- - V$)**

$$\frac{\partial V}{\partial t} + \mathcal{L}_{SXh} V = -\frac{h}{1-R}\Delta V + f^{H,D} V + [(r^R + b^{D,C_0} - f^{H,D}) R^{C_0} + (c^D + b^{D,C_0} - f^{H,D}) M^{C_0}] X^{D,C_0},$$

$$\mathcal{L}_{SXh} = \frac{1}{2}\sum_{i,k=1}^{N} \rho^{S^i S^k} \sigma^{S^i} \sigma^{S^k} S^i S^k \frac{\partial^2}{\partial S^i \partial S^k} + \frac{1}{2}\sum_{j,l=0}^{N} \rho^{X^j X^l} \sigma^{X^j} \sigma^{X^l} X^j X^l \frac{\partial^2}{\partial X^j \partial X^l} + \sum_{i=1}^{N}\sum_{j=0}^{N} \rho^{S^i X^j} \sigma^{S^i} \sigma^{X^j} S^i X^j \frac{\partial^2}{\partial S^i \partial X^j}$$
$$+ \frac{1}{2}(\sigma^h)^2 \frac{\partial^2}{\partial h^2} \sum_{i=1}^{N} \rho^{S^i h} \sigma^{S^i} \sigma^h S^i \frac{\partial^2}{\partial S^i \partial h} + \sum_{j=0}^{N} \rho^{X^j h} \sigma^{X^j} \sigma^h X^j \frac{\partial^2}{\partial X^j \partial h}$$
$$+ \sum_{i=1}^{N} (r^i - q^i - \rho^{S^i X^i} \sigma^{S^i} \sigma^{X^i}) S^i \frac{\partial}{\partial S^i} + \sum_{j=0}^{N} (r^D - r^j) X^j \frac{\partial}{\partial X^j} + (\mu^h - M^h \sigma^h) \frac{\partial}{\partial h}$$

**Nonlinear ($M = V$) and linear ($M = W$) pricing PDEs**

$$M = V \quad \Rightarrow \quad \frac{\partial V}{\partial t} + \mathcal{L}_{SXh} V - fV = (\bar{r} R^{C_0} + \bar{m} M^{C_0}) X^{D,C_0} + h(V)^+,$$

$$M = W \quad \Rightarrow \quad \frac{\partial V}{\partial t} + \mathcal{L}_{SXh} V - \left(\frac{h}{1-R} + f\right) V = (\bar{r} R^{C_0} + \bar{m} M^{C_0}) X^{D,C_0} + h(W)^+ - \frac{h}{1-R} W$$

# PDE problems for XVA

## XVA value

$U = V - W$

## Final condition

$W(T, S, \bar{X}) = V(T, S, X, h) = \text{Payoff}(S, X) \quad \Rightarrow \quad U(T, S, X, h) = 0$

## PDE problems

- Nonlinear final value problem ($M = V$):

$$\begin{cases} \frac{\partial U}{\partial t} + \mathcal{L}_{SXh} U - fU = h(W + U)^+ + (\bar{r} R^{C_0} + \bar{m} M^{C_0}) X^{D, C_0}, \\ U(T, S, X, h) = 0; \end{cases}$$

- Linear final value problem ($M = W$):

$$\begin{cases} \frac{\partial U}{\partial t} + \mathcal{L}_{SXh} U - \left( \frac{h}{1-R} + f \right) U = h(W)^+ + (\bar{r} R^{C_0} + \bar{m} M^{C_0}) X^{D, C_0}, \\ U(T, S, X, h) = 0. \end{cases}$$

In both cases, $(t, S, X, h) \in [0, T) \times (0, +\infty)^N \times (0, +\infty)^{N+1} \times (0, +\infty)$.

# Formulation in terms of expectations

In order to compute the values of $U$ by using the Monte Carlo method, we apply the Feynman-Kac theorem to formulate the PDE problems in terms of expectations.

## Nonlinear case ($M = V$)

$$U(t,S,X,h) = E_t^Q\Big[-\int_t^T e^{-f(u-t)}\Big(h_u(W(u,S_u,\bar{X}_u) + U(u,S_u,X_u,h_u))^+ + (\bar{r}R_u^{C_0} + \bar{m}M_u^{C_0})X_u^{D,C_0}\Big)du \,|\, S_t = S, X_t = X, h_t = h\Big]$$

## Linear case ($M = W$)

$$U(t,S,X,h) = E_t^Q\Big[-\int_t^T e^{-\int_t^u (\frac{h_r}{1-R}+f)dr}\Big(h_u(W(u,S_u,\bar{X}_u))^+ + (\bar{r}R_u^{C_0} + \bar{m}M_u^{C_0})X_u^{D,C_0}\Big)du \,|\, S_t = S, X_t = X, h_t = h\Big]$$

# Formulation in terms of expectation - nonlinear case

## Nonlinear case ($M = V$)

$$U(0,S,X,h) = E_0^{Q^D}\Bigg[-\int_0^T e^{-fu}\Big(h_u\Big(W(u,S_u,\bar{X}_u) + U(u,S_u,X_u,h_u)\Big)^+ + \Big(\bar{r}R_u^{C_0} + \bar{m}M_u^{C_0}\Big)X_u^{D/C_0}\Big)du \,\Bigg|\, S_0 = S, X_0 = X, h_0 = h\Bigg].$$

## Fixed point iteration

We set $U^0 = 0$ and recursively compute:

$$U^{\ell+1}(0,S,X,h) = E_0^{Q^D}\Bigg[-\int_0^T e^{-fu}\Big(h_u\Big(W(u,S_u,\bar{X}_u) + U^{\ell}(u,S_u,X_u,h_u)\Big)^+ + \Big(\bar{r}R_u^{C_0} + \bar{m}M_u^{C_0}\Big)X_u^{D/C_0}\Big)du \,\Bigg|\, S_0 = S, X_0 = X, h_0 = h\Bigg]$$

for $\ell = 0, 1, 2, \ldots$ until convergence is attained.

# Numerical examples

## Financial data

| | | | |
|---|---|---|---|
| $r = (0.07, 0.09, 0.12)$ | $\sigma^S = (0.30, 0.20)$ | $q = (0.07, 0.08)$ | $r^D = 0.06$ |
| $X_0 = (0.13, 0.89, 1.12)$ | $\sigma^X = (0.38, 0.40, 0.35)$ | $R_0^{C_0} = 25$ | $M_0^{C_0} = 25$ |
| $h_0 = 0.20$ | $\kappa = 0.01$ | $\sigma^h = 0.2$ | $R = 0.3$ |
| $c^D = 0.06$ | $r^R = 0.05$ | $b^{D,C_0} = 0.02$ | $f = 0.06$ |
| $K^1 = 12$ | $K^2 = 15$ | $K = 5$ | |

## Payoff

- Best of put/put option $\Rightarrow$ $G(t, S_t^1, S_t^2, X_t^{D,C_1}, X_t^{D,C_2}) = \max((K^1 - S_t^1 X_t^{D,C_1})^+, (K^2 - S_t^2 X_t^{D,C_2})^+)$
- Basket call option $\Rightarrow$ $G(t, S_t^1, S_t^2, \bar{X}_t) = \left( \sum_{i=1}^{N} \alpha^i S_t^i X_t^{D,C^i} - K \right)^+$

For more numerical examples see:
[6] R. Simonella and C. Vázquez. *XVA in a multi-currency setting with stochastic foreign exchange rates.*

# Best of put/put option

**Payoff**

$$G(t, S_t^1, S_t^2, X_t^{D,C_1}, X_t^{D,C_2}) = \max((K^1 - S_t^1 X_t^{D,C_1})^+, (K^2 - S_t^2 X_t^{D,C_2})^+)$$

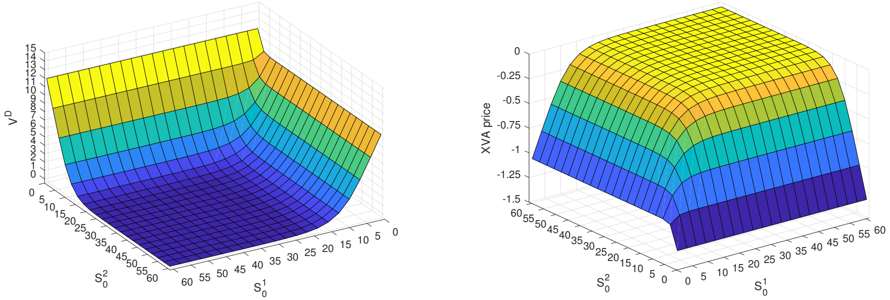


Figure: Risky price and XVA price in the nonlinear case

# Best of put/put option

## Payoff

$$G(t, S_t^1, S_t^2, X_t^{D,C_1}, X_t^{D,C_2}) = \max((K^1 - S_t^1 X_t^{D,C_1})^+, (K^2 - S_t^2 X_t^{D,C_2})^+)$$

| $\sigma^X$ | XVA price |
|---|---|
| (0.275,0.05,0.05) | [-0.2039,-0.1952] |
| (0.275,0.05,0.50) | [-0.3027,-0.2900] |
| (0.275,0.50,0.05) | [-0.2898,-0.2766] |
| (0.275,0.50,0.50) | [-0.3710,-0.3563] |
| (0.000,0.00,0.00) | [-0.2000,-0.1914] |

Table: XVA price confidence intervals in the nonlinear case for different sets of FX rates volatilities values

# Basket call option

## Payoff

$$G(t, S_t^1, S_t^2, \bar{X}_t) = \left( \sum_{i=1}^{N} \alpha^i S_t^i X_t^{D,C^i} - K \right)^+$$

| N | $V$ | XVA | Time | FPI |
|---|---|---|---|---|
| 2 | [2.5433,2.6296] | [-0.3325,-0.3215] | 2.0698 | 11 |
| 4 | [3.6617,3.7444] | [-0.5197,-0.5074] | 3.3307 | 11 |
| 8 | [0.0323,0.0605] | [-0.4720,-0.4682] | 7.1890 | 9 |
| 16 | [1.1801,1.2040] | [-0.3061,-0.3020] | 13.3564 | 11 |

Table: Risky price and XVA price confidence intervals, elapsed time and number of fixed point interations (FPI) in the nonlinear case for increasing number of underlying assets

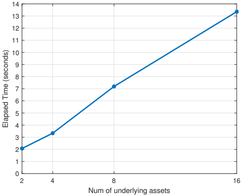


Figure: Elapsed time in the nonlinear case

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