Dipartimento di Matematica,
Università di Bologna

An introduction to rating triggers for collateral-inclusive XVA in an ICTMC framework

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Data inputs

$$\left(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}\right)$$

$$\left(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q}^h\right)$$

$$(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$$

Data inputs

$(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ ← Input

$(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q}^h)$

From/To	F1+	F1	F2	F3	B	C	D	Withdrawal
F1+	86.950	6.350	0.550	0.090	0.040	0.000	0.050	5.970
F1	2.720	85.850	6.200	0.500	0.290	0.000	0.050	4.390
F2	0.260	3.180	84.420	4.660	1.370	0.100	0.090	5.920
F3	0.190	0.400	8.250	75.270	6.540	0.190	0.230	8.930
B	0.000	0.030	0.460	4.040	81.890	2.780	0.950	9.850
C	0.000	0.000	0.000	0.000	31.880	42.400	10.440	15.280
D	0.000	0.000	0.000	0.000	0.000	0.000	100.000	0.000

1. "Withdrawal" means the percentage of companies, who decided to withdraw from being rated by this rating agency;
2. Almost tri-diagonal.

$(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$

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F1	0.741
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Calibration

$(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$

1. Only data for default column;
2. Default probabilities are usually higher under $\mathbb{Q}$, than $\mathbb{P}$.

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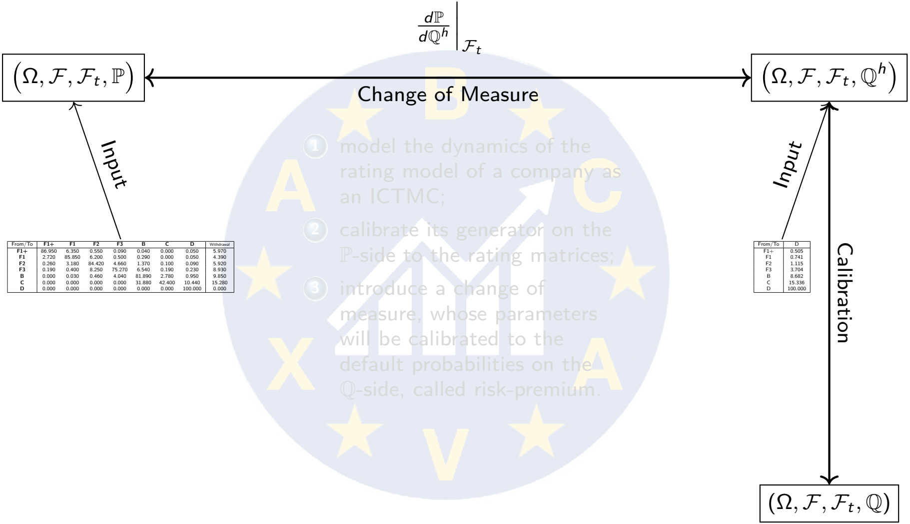

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2. calibrate its generator on the $\mathbb{P}$-side to the rating matrices;
3. introduce a change of measure, whose parameters will be calibrated to the default probabilities on the $\mathbb{Q}$-side, called risk-premium.

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$$\left.\frac{d\mathbb{P}}{d\mathbb{Q}^h}\right|_{\mathcal{F}_t}$$

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ICTMC X_t

Describes the evolution of a company's rating

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Can't use rating matrices directly

Calibration under $\mathbb{P}$

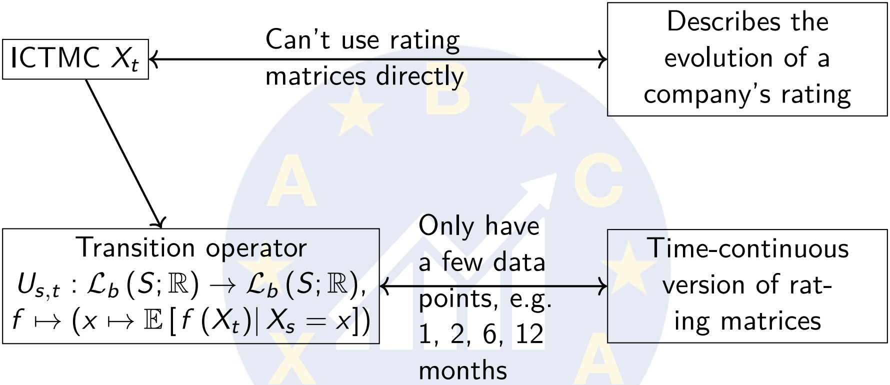
ICTMC X_t

Can't use rating matrices directly

Describes the evolution of a company's rating

Transition operator
$U_{s,t} : \mathcal{L}_b(S;\mathbb{R}) \to \mathcal{L}_b(S;\mathbb{R})$,
$f \mapsto (x \mapsto \mathbb{E}[f(X_t) | X_s = x])$

Calibration under $\mathbb{P}$



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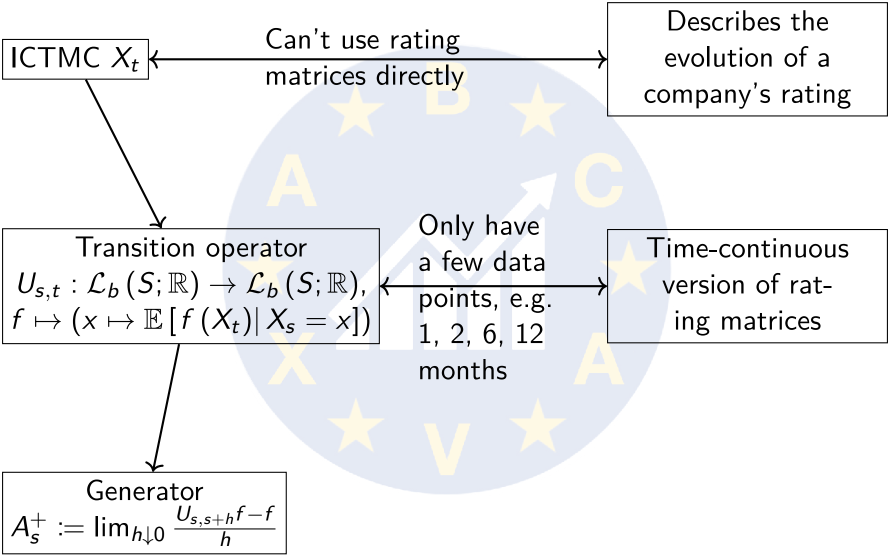

Calibration under $\mathbb{P}$

ICTMC X_t ⟷ (Can't use rating matrices directly) ⟷ Describes the evolution of a company's rating

↓

Transition operator
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⟷ (Only have a few data points, e.g. 1, 2, 6, 12 months) ⟷ Time-continuous version of rating matrices

↓

Generator
$A_s^+ := \lim_{h\downarrow 0} \frac{U_{s,s+h}f - f}{h}$

⟷ (Have to extract generators from data) ⟷ If known, and X_t is homogeneous, we have $U_t = \exp(At)$

Rating model

Goal and main theorem

Theorem

Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}\right)$ *be a filtered probability space with a filtration* $\mathcal{F}_t$ *satisfying the usual conditions,* $T > 0$ *a finite time horizon and* $S = \{1, \dots, K\}$*,* $K \in \mathbb{N}$*, a finite state space with Borel-*σ*-algebra* $\mathcal{S}$*.*

Rating model

Goal and main theorem

Theorem

Let $X_\cdot$ denote an ICTMC with respect to $\mathcal{F}_t$ with extended right-generator A_t^+ and domain $\mathrm{dom}\,(A)$. We get for a positive and "good" function h, that the probability measure $\mathbb{Q}^h$ defined by $\frac{d\mathbb{Q}_t^h}{d\mathbb{P}_t} = L_t^h$, where

$$L_t^h := \frac{h(t, X_t)}{h(0, X_0)} \exp\left(- \int_0^t \frac{\partial_t^+ h(s, X_s) + \left(A_s^{+,\mathbb{P}} h(s, \cdot)\right)(X_s)}{h(s, X_s)} ds \right)$$

is well-defined and equivalent to $\mathbb{P}$.

Rating model

Goal and main theorem

Theorem

Moreover, the process X. is an ICTMC under $\mathbb{Q}^h$ *as well with extended right-generator* $A_t^{+,h}$, $i,j = 1,\dots,K$,

$$A_{ij}^{+,h}(t) = \begin{cases} A_{ij}^{+,\mathbb{P}}(t)\,\frac{h_j(t)}{h_i(t)}, & i \neq j, \\ -\sum_{k\neq i} A_{ij}^{+,\mathbb{P}}(t)\,\frac{h_j(t)}{h_i(t)}, & i = j, \end{cases} \tag{1.1}$$

such that $\operatorname{dom}\left(A^{+,\mathbb{P}}\right) = \operatorname{dom}\left(A^{+,h}\right)$ *and the functions* $h(t,x)$ *have been identified with time-dependent vectors* $h_i(t) \in \mathbb{R}^K_{>0}$.

Calibration under $\mathbb{Q}$

Exponential change of measure

We have the following result by Palmowski and Rolski (2002) for homogeneous Markov processes.

Theorem

Let $X.$ denote a Markov process with Borel state space with respect to $\mathcal{F}_t$ with extended generator A and domain $\operatorname{dom}(A)$. We get for a positive and "good" function h, that the probability measure $\mathbb{Q}^h$ defined by $\frac{d\mathbb{Q}_t^h}{d\mathbb{P}_t} = L_t^h$, where

$$L_t^h := \frac{h(X_t)}{h(X_0)} \exp\left(-\int_0^t \frac{(Ah(\cdot))(X_s)}{h(X_s)} ds\right)$$

is well-defined and equivalent to $\mathbb{P}$. Then, $X.$ is under $\mathbb{Q}^h$ a Markov process as well with extended generator A^h and $\operatorname{dom}(A) = \operatorname{dom}\left(A^h\right)$.

Calibration under $\mathbb{Q}$

Time-Space state space

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Therefore, consider the new state-space $S^T := [0, T] \times S$ with Borel-σ-algebra $\mathcal{S}^T := \mathcal{B}([0, T]) \vee \mathcal{S}$.

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In light of Kolmogorov's theorem to construct a Markov process, we need a new sample-space as well, namely $\Omega^T := [0, T] \times \Omega$ with σ-algebra $\mathcal{F}^T := \mathcal{B}([0, T]) \vee \mathcal{F}$.

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We can now define the new time-space process $X_\cdot^T$ for $t \in [0, T]$ by

$$X_t^T(\omega^T) := X_t^T((s, \omega)) := (s + t, X_{s+t}(\omega))$$

with the probability measure defined for $E^T \in \mathcal{S}^T$ and $x^T \in [0, T] \times S$ by

$$\mathbb{P}_{x^T}^T \left[E^T\right] = \mathbb{P}^T \left[E^T \middle| X_0^T = x^T\right] := \mathbb{P}\left[E_s \mid X_s = x\right],$$

where $E_s = \left\{x \in S : (s, x) \in E^T\right\} \subseteq \mathcal{S}^T$ is the time slice.

Calibration under $\mathbb{Q}$

Identification of the generator

The last step in showing our theorem is an identification for the time-space generator in terms of its inhomogeneous version.

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In a recent paper by Köpfer and Rüschendorf (2021) this was solved by showing that the following definition of right-generators is equivalent to our case.
A family of operators $\left(A_t^+\right)_{t\in[0,T)}$ on $\left(\mathcal{L}_b\left(S\right), \|\cdot\|_\infty\right)$ is said to be a right generator of a Markov process X. if for all $f \in \mathcal{D}_+\left(A\right)$, for all $x \in S$ and for all $s \leq t < T$ $(0 < s \leq t)$ it holds

$$\partial_t^+ \mathbb{E}^{\mathbb{P}}\left[f\left(t, X_t\right) \middle| X_s = x\right] = \mathbb{E}^{\mathbb{P}}\left[\partial_t^+ f\left(t, X_t\right) + A_t^+ f(t,\cdot)(X_t) \middle| X_s = x\right],$$

respectively, and all limits are regarded as pointwise.

Market data
Reminder

There are two things we have to consider

1. Remove the withdrawal;
2. Extract a generator;

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Market data

Removing the withdrawal

Input : $R_t^M \in \mathbb{R}^{K,K}$ with row-sums less or equal to 1
Output: $R_t^A \in \mathbb{R}^{K,K}$ with row-sums equal to 1

for $i \leftarrow 1$ **to** K **do**
  $\mathrm{wd} \leftarrow \sum_{j=1}^{K} \left(R_t^M\right)_{ij}$;
  if $\mathrm{wd} > 0$ **then**
    $\mathrm{y} \leftarrow \left(R_t^M\right)_{i,j=1,\ldots,K}$;
    $\mathrm{y}\,(\mathrm{y} == 0) \leftarrow 1e^{-10}$;
    $\mathrm{b} \leftarrow \frac{\mathrm{y}}{\sum_{j=1}^{K} \mathrm{y}_j} \cdot (1 - \mathrm{wd})$;
    $\left(R_t^A\right)_{i,j=1,\ldots,K} \leftarrow \left(R_t^M\right)_{i,j=1,\ldots,K} + \mathrm{b}$
  else
    $\left(R_t^A\right)_{i,j=1,\ldots,K} \leftarrow \left(R_t^M\right)_{i,j=1,\ldots,K}$
  end
end

Algorithm 1: Adjustment of the market rating matrices.

Market data

Extracting generator

In general, this is still an open problem and called *embedding problem*. We decided to apply the following algorithm whenever a non-valid generator occured.

Input : $R_t^A \in \mathbb{R}^{K,K}$
Output: $A_t^M \in \mathbb{R}^{K,K}$ approximated generator of R_t^A

$\mathrm{A} \leftarrow \texttt{logm}\left(R_t^A\right)$;
$\mathrm{A}\left(\mathrm{A}<0\right) \leftarrow 0$;
for $i \leftarrow 1$ **to** K **do**
  $\left(\mathrm{A}\right)_{ii} \leftarrow -\sum_{j\neq i}\left(\mathrm{A}\right)_{ij}$;
end
$A_t^M \leftarrow \mathrm{A}$;

Algorithm 2: Approximation of the market generators.

Our Model: PHCTMC

Let $T_0 \in [0, T]$ be the initial point and T_k, $k = 1, \ldots, n$ be the points in time when rating matrices are available in an increasing order with $T_n = T$, then $X.$ is assumed to be homogeneous on each $[T_k, T_{k+1})$ $k = 0, \ldots, n-1$.

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Now, by the Chapman-Kolmogorov equation we get

$$U^{\mathbb{P}}_{T_0,T} = U^{\mathbb{P}}_{T_0,T_1} \cdot U^{\mathbb{P}}_{T_1,T_2} \cdots U^{\mathbb{P}}_{T_{n-1},T_n} = \prod_{k=1}^{n} U^{\mathbb{P}}_{T_{k-1},T_k}. \tag{1.2}$$

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By homogeneity on each sub-interval we know that the evolution system will reduce to a semigroup and its generator will be time-constant with an explicit formula

$$U^{\mathbb{P}}_{T_{k-1},t} = U^{\mathbb{P}}_{t-T_{k-1}} = \exp\left(A^{\mathbb{P}}_k \left(t - T_{k-1}\right)\right), \quad t \in [T_{k-1}, T_k). \tag{1.3}$$

Our Model: PHCTMC

Hence, to extract these generators from the market data, we denote by R_k^M the rating matrices at times T_k, $k = 1, \dots, n$ and solve

$$U^{\mathbb{P}}_{T_0, T_k} \overset{!}{=} R_k^M,$$

which is by (1.3) under the assumption that $U^{\mathbb{P}}_{T_0, T_{k-1}}$ is invertible and $\left(U^{\mathbb{P}}_{T_0, T_{k-1}}\right)^{-1} \cdot R_k^M$ has a matrix logarithm equivalent to

$$A_k^{\mathbb{P}} = \frac{\log\left(\left(U^{\mathbb{P}}_{T_0, T_{k-1}}\right)^{-1} \cdot R_k^M\right)}{T_k - T_{k-1}}, \qquad U^{\mathbb{P}}_{T_0, T_{k-1}} = \prod_{l=1}^{k-1} U^{\mathbb{P}}_{T_{l-1}, T_l}.$$

Calibration

Procedure

$$\min_{\substack{h_k \in \mathbb{R}^{K,1}_{>0}, (h_k)_K = 1, \\ A_k \in \mathcal{A}}} \left\| U^{\mathbb{Q}^L}_{T_0, T_{k-1}} \cdot \exp\left(A^h_k \left(T_k - T_{k-1} \right) \right) \cdot e_K - \mathrm{PD}\left(T_k \right) \right\|_{\mu_{\mathbb{Q}}},$$

$$+ \left\| A_k - A^{\mathbb{P}}_k \right\|_{\mu_{\mathbb{P}}}$$

$$\mathcal{A} := \Big\{ A \in \mathbb{R}^{K,K} : \text{ for all } i, j = 1, \ldots, K$$

$$A_{K,j} = 0, \; A_{i,j} \geq 0 \text{ for } i \neq j \text{ and } A_{i,i} \leq 0 \Big\}.$$

$$A^h_k := \begin{cases} A_k \frac{(h_k)_j}{(h_k)_i}, & i \neq j, \\ -\sum_{k \neq i} A_k \frac{(h_k)_j}{(h_k)_i}, & i = j, \end{cases}$$

$$U^{\mathbb{Q}^h}_{T_0, T_{k-1}} = \prod_{l=1}^{k-1} U^{\mathbb{Q}^h}_{T_{l-1}, T_l}.$$

Calibration

Method from literature

In the literature it is proposed that one retrieves the generator under $\mathbb{Q}$ first and calibrates to the probability of default afterwards. This leads to

From \ To	F1+	F1	F2	F3	B	C	D
F1+	0.000 %	98.047 %	0.000 %	0.000 %	0.838 %	0.351 %	0.764 %
F1	0.000 %	97.033 %	0.000 %	0.000 %	1.468 %	0.696 %	0.802 %
F2	0.000 %	89.720 %	0.000 %	0.000 %	5.570 %	2.860 %	1.850 %
F3	0.000 %	85.545 %	0.000 %	0.000 %	7.724 %	4.049 %	2.683 %
B	0.000 %	1.868 %	0.000 %	0.000 %	56.863 %	32.511 %	8.758 %
C	0.000 %	0.044 %	0.000 %	0.000 %	13.455 %	71.167 %	15.334 %
D	0.000 %	0.000 %	0.000 %	0.000 %	0.000 %	0.000 %	100.000 %

Calibration

Our model

In our model with the additional calibration under $\mathbb{P}$ we get a much better result:

From \ To	F1+	F1	F2	F3	B	C	D
F1+	92.470 %	6.753 %	0.585 %	0.096 %	0.043 %	0.001 %	0.053 %
F1	2.845 %	89.782 %	6.485 %	0.523 %	0.305 %	0.008 %	0.053 %
F2	0.276 %	3.380 %	89.732 %	4.954 %	1.456 %	0.105 %	0.096 %
F3	0.209 %	0.439 %	9.059 %	82.653 %	7.181 %	0.206 %	0.252 %
B	0.007 %	0.034 %	0.513 %	4.522 %	90.826 %	3.046 %	1.053 %
C	0.003 %	0.004 %	0.076 %	0.989 %	37.477 %	49.160 %	12.292 %
D	0.000 %	0.000 %	0.000 %	0.000 %	0.000 %	0.000 %	100.000 %

Calibration

Results

Error \ Time	Fitch's data				S&P's data
	$t=\frac{1}{12}$	$t=\frac{3}{12}$	$t=\frac{6}{12}$	$t=1$	$t=1$
Calibration Errors					
`fmincon`	1.56e-07	9.05e-08	6.87e-08	6.37e-08	0.00136
$\frac{1}{K^2}\left\|\left\|R_t^{\mathbb{P}}-R_t^A\right\|\right\|_{\mathbb{R}^{K,K}}$	2.69e-06	2.35e-05	0.000101	0.000464	1.48e-05
$\frac{1}{K}\left\|\left\|R_t^{\mathbb{Q}}e_K-\mathrm{PD}(t)\right\|\right\|_{\mathbb{R}^K}$	1.64e-07	1.59e-08	2.18e-09	5.83e-09	0.00204

The total computational time for the calibration was 4.68 seconds with Fitch's data and 15 seconds with S&P's data.

Simulation

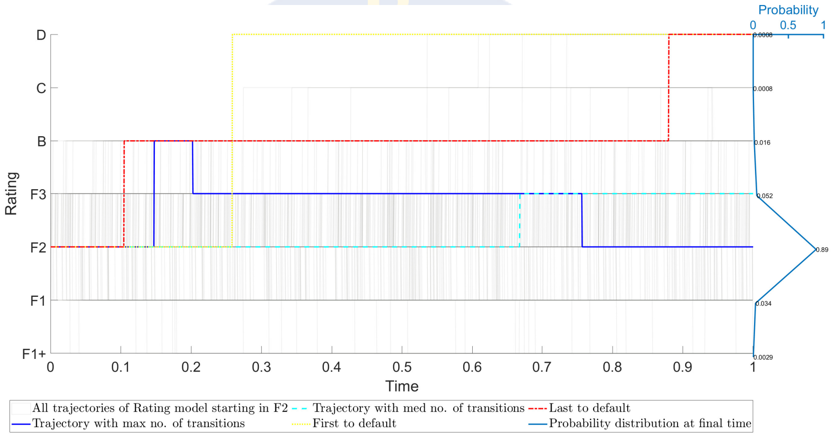
For the simulation of the piecewise-homogeneous CTMC we are iterating over all intervals, where the process is homogeneous and use *Gillespie's stochastic simulation algorithm (SSA)* or *Kinetic Monte Carlo (KMC)* on each of them.

	Fitch's data				S&P's data
Error \ Time	$t = \frac{1}{12}$	$t = \frac{3}{12}$	$t = \frac{6}{12}$	$t = 1$	$t = 1$
Simulation Errors, $M = 10000$ simulations					
$\frac{1}{K^2} \left\| R_t^{\mathbb{P}} - R_t^{\text{Sim},\mathbb{P}} \right\|_{\mathbb{R}^{K,K}}$	0.000354	0.000671	0.000762	0.000952	0.000773
$\frac{1}{K^2} \left\| R_t^{\mathbb{Q}} - R_t^{\text{Sim},\mathbb{Q}} \right\|_{\mathbb{R}^{K,K}}$	0.000392	0.000923	0.000664	0.00106	0.000881

For S&P's data the SSA took 7.66 seconds and it took roughly 2.98 seconds using Fitch's data. We calculated the SSAs for each initial rating in parallel on a CPU.

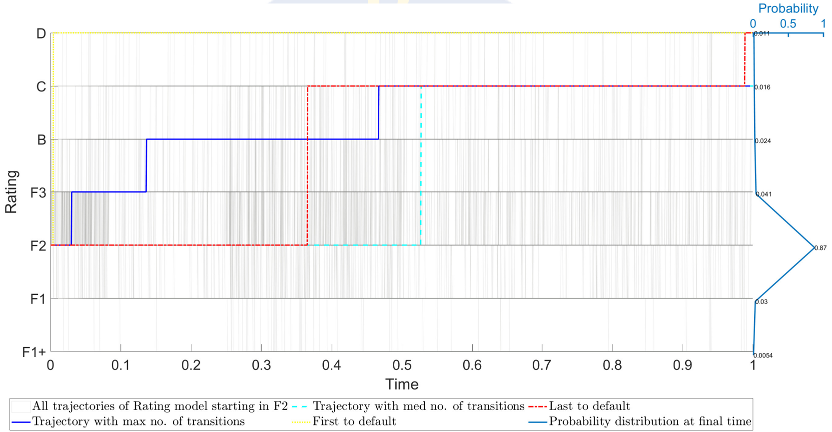
Simulation

Simulated trajectories of CTMC X. calibrated to Fitch's data set starting in rating F2 under measure $\mathbb{P}$.



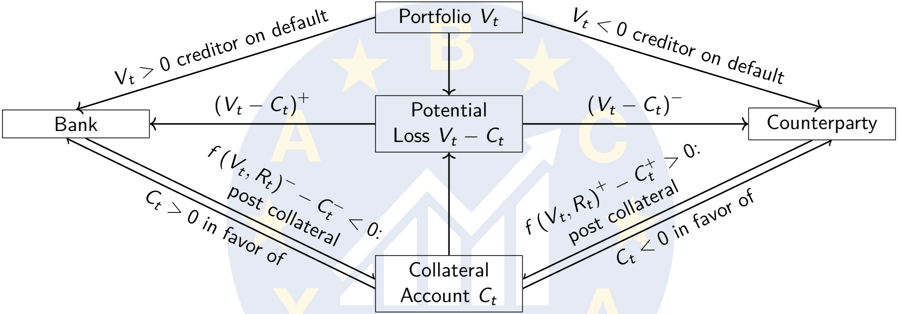
Simulation

Simulated trajectories of CTMC X. calibrated to Fitch's data set starting in rating F2 under measure $\mathbb{Q}$.



Rating Triggers

Overview



1. V_t denotes the portfolio;
2. C_t denotes the collateral account;
3. R_t denotes the rating process;
4. f stands for the different collateral agreements;

Rating Triggers

Collateral Account

The collateral postings of the bank are defined as

$$\mathbb{1}_{\left|\left(\mathrm{V}_{t_j}+\rho^B(R^B_{t_j})\right)^- - C^-_{t_j^-}\right|>m} \cdot \left(\left(\mathrm{V}_{t_j}+\rho^B(R^B_{t_j})\right)^- - C^-_{t_j^-}\right) =: \mathrm{CP}^{\mathsf{B}}_{t_j}.$$

For the counterparty we have analogously

$$\mathbb{1}_{\left|\left(\mathrm{V}_{t_j}-\rho^C(R^C_{t_j})\right)^+ - C^+_{t_j^-}\right|>m} \cdot \left(\left(\mathrm{V}_{t_j}-\rho^C(R^C_{t_j})\right)^+ - C^+_{t_j^-}\right) =: \mathrm{CP}^{\mathsf{C}}_{t_j},$$

where e.g. $\rho^x(i) := \sum_{j=1}^K r_j^x \mathbb{1}_j(i)$, $r_i^x \in \mathbb{R}_{\geq 0}$, $x \in \{B, C\}$, $i = 1, \dots, K$.
In total, we have

$$C_{t_0} := 0, \quad C_{t_n} := 0, \quad C_{u^-} := C_{\beta(u)}$$

where $\beta(u)$ is the last update before u and $t_0 < u \leq t_n$ and

$$C_{t_j} := C_{t_j^-} + \mathrm{CP}^{\mathsf{B}}_{t_j} + \mathrm{CP}^{\mathsf{C}}_{t_j}.$$

Rating Triggers

Agreements

We will discuss the following three scenarios of collateral agreements:

1. In the case of no collateral $C_t \equiv 0$ we call the agreement *uncollateralized*;
2. In the case $C_{t_j} := V_{t_j}$ and $j = 1, \ldots, n$, $n \in \mathbb{N}$, we call it *perfectly collateralized*, if it is independent of the rating processes, e.g. $f(V_t, R_t) := V_t$, $t_j \in [0, T]$;
3. Otherwise, if it is dependent on the rating processes, we call the agreement *collateralization with rating tiggers*.

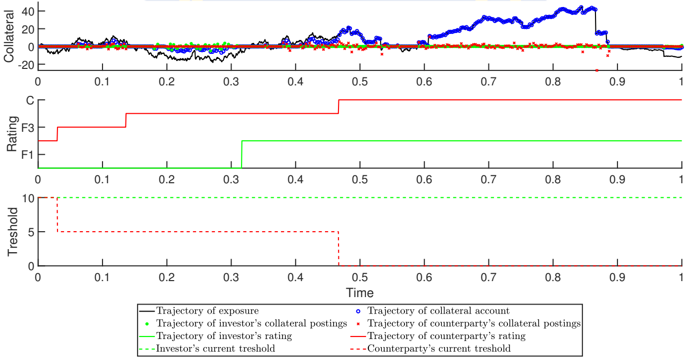
CXVA with different collateral agreements

Rating thresholds for both counterparties using Fitch's data.

F1+	F1	F2	F3	B	C	D
Bank						
10	10	10	5	5	0	0
Counterparty						
10	10	10	5	5	0	0

CXVA with different collateral agreements

One trajectory of a collateral agreement with rating triggers using Fitch's data. The top picture shows the collateral account and portfolio over time, the middle one the rating evolution and the bottom one the corresponding rating thresholds.



CXVA with different collateral agreements

CXVA with the different collateral agreements (uncollateralized, perfectly collateralized and rating triggers) using Fitch's data and $\mathrm{LGD}_B = 0.6$, as well as $\mathrm{LGD}_C = 0.6$ with $M = 10000$ simulations and thresholds as above.

CXVA	Uncollateralized	Rating Triggers	Perfectly collateralized
CDVA	0.0638	0.0351	0.0133
CCVA	0.0842	0.0438	0.0147
CBVA	−0.0204	−0.00873	−0.00139

where we define without rehypothecation[1]

$$\mathrm{CBVA}(t, T, \mathrm{C}) := \mathrm{CDVA}(t, T, \mathrm{C}) - \mathrm{CCVA}(t, T, \mathrm{C})$$

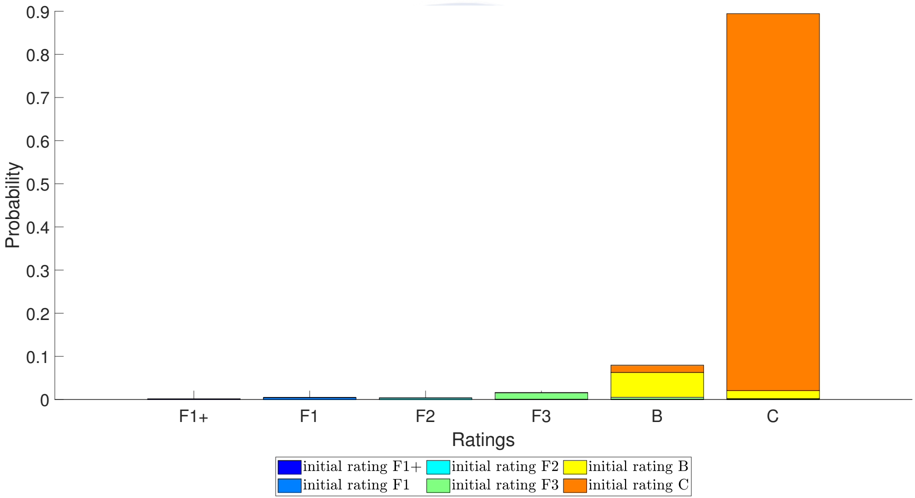
$$\mathrm{CDVA}(t, T, \mathrm{C}) := -\mathbb{E}\left[\mathbb{1}_{\tau=\tau_B<T}\mathrm{LGD}_B\left(V_\tau^- - C_\tau^-\right)^- \Big| \mathcal{G}_t\right]$$

$$\mathrm{CCVA}(t, T, \mathrm{C}) := \mathbb{E}\left[\mathbb{1}_{\tau=\tau_C<T}\mathrm{LGD}_C\left(V_\tau^+ - C_\tau^+\right)^+ \Big| \mathcal{G}_t\right]$$

[1] $X^+ = \max(X, 0)$ and $X^- = \min(X, 0)$

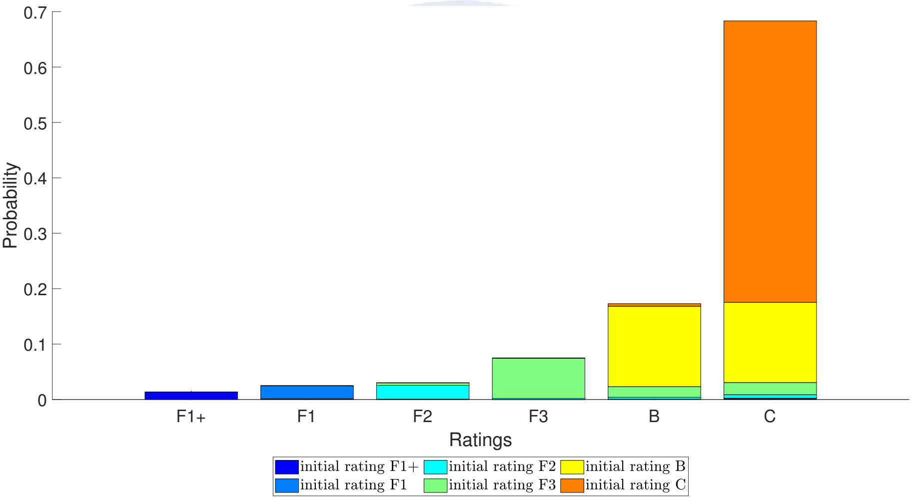
Pre-default distribution

Fitch's data under $\mathbb{P}$



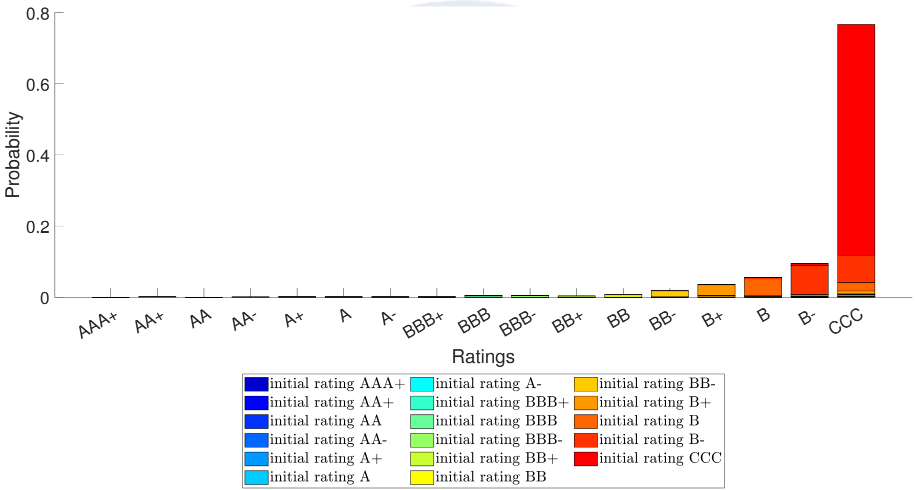
Pre-default distribution

Fitch's data under $\mathbb{Q}$



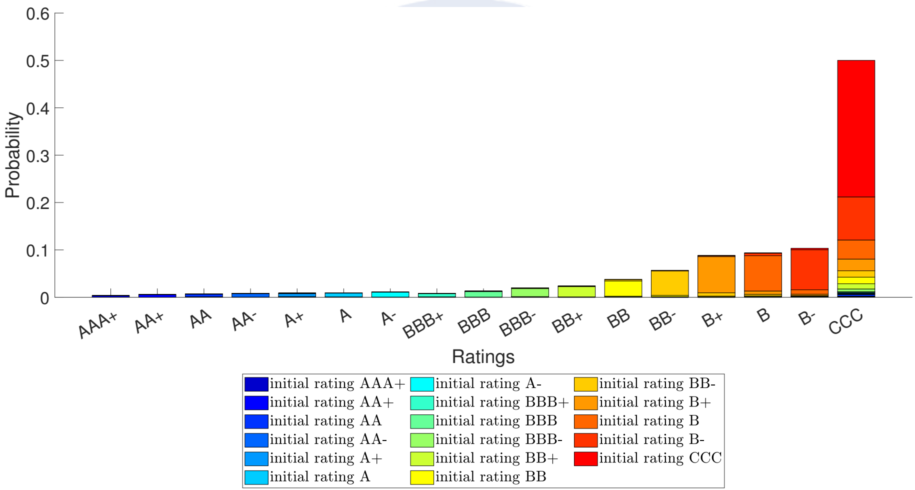
Pre-default distribution

S&P's data under $\mathbb{P}$



Pre-default distribution

S&P's data under $\mathbb{Q}$



Current research

1. We found a new set of data and are trying to derive our own rating matrices without imperfections;
2. Include correlations between ratings and interest rates.

Thank you for your attention!

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